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Bases in coset conformal field theory from AGT correspondence and Macdonald polynomials at the roots of unity

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Abstract

We continue our study of AGT correspondence between instanton counting on $\mathbb{C}^2/\mathbb{Z}_p$ and conformal field theories with the symmetry algebra $\mathcal{A}(r,p)$. In the cases $r=1, p=2$ and $r=2, p=2$ this algebra specialized to: $\mathcal{A}(1,2)=\mathcal{H}\oplus\widehat{\mathfrak{sl}}(2)_1$ and $\mathcal{A}(2,2)=\mathcal{H}\oplus\widehat{\mathfrak{sl}}(2)_2\oplus\mathsf{NSR}$.

As the main tool we use the a construction of the algebra $A(r,2)$ as the limit of the toroidal $\mathfrak{gl}(1)$ algebra for q,t tend to -1. We claim that the basis of the representation of the algebra $\mathcal{A}(r,2)$ (or equivalently, of the space of the local fields of the corresponding CFT) can be expressed through Macdonald polynomials with the parameters q,t tend to -1. The vertex operator which naturally arises in this construction has factorized matrix elements in this basis. We also argue that the singular vectors of $\mathcal{N}=1$ Super Virasoro algebra can be realized in terms of Macdonald polynomials for rectangular Young diagram and parameters q,t tend to -1.

1 Introduction

1.1. This paper is a sequel to [1]. We study some questions concerning conformal field theory (CFT) using (mainly for motivation) the AGT relation ([2]). Namely, the AGT relation between Conformal field theory with the symmetry algebra $\mathcal{A}$ and Instanton moduli space $\mathcal{M}$ suggests the following structure:

- To every torus fixed point $p\in\mathcal{M}$ corresponds basic vector $v_p\in\pi_{\mathcal{A}}$ (highest weight representation of the algebra $\mathcal{A}$).
- Basis v_p is orthogonal under the natural product and the norm of the vector v_p equals the determinant of the torus action in the tangent space of p.
- Matrix elements of geometrically defined vertex operators have completely factorized form. The last expressions are also denoted by Z_{bif} (contribution of the bifundamental multiplet).

- There is a commutative algebra (Integrals of Motion) which is diagonalized in the basis v_p.

For more details and motivation of this structure see [3], [4] and [1]. In this paper we construct this basis for certain conformal filed theories.

1.2. It was suggested in papers [5],[6] that CFT with the symmetry

$$\mathcal{A}(r,p) \stackrel{\text{def}}{=} \mathcal{H} \times \widehat{\mathfrak{sl}}(p)_r \times \frac{\widehat{\mathfrak{sl}}(r)_p \times \widehat{\mathfrak{sl}}(r)_{n-p}}{\widehat{\mathfrak{sl}}(r)_n} \tag{1.1}$$

corresponds to the instanton counting on $\mathbb{C}^2/\mathbb{Z}_p$, where $\mathbb{Z}_p$ acts by formula $(z_1, z_2) \mapsto (\omega z_1, \omega^{-1} z_2)$, $\omega = \exp(\frac{2\pi i}{p})$. The quotient $\mathbb{C}^2/\mathbb{Z}_p$ is singular, and there are several possibilities to define instanton moduli space. One of them can be performed as follows. Denote by $\mathcal{M}(r,N)$ the smooth compactified moduli space of $U(r)$ instantons on $\mathbb{C}^2$ with the topological number N. The set $\mathcal{M}(r,N)^{\mathbb{Z}_p}$ of $\mathbb{Z}_p$-invariant points on $\mathcal{M}(r,N)$ is a smooth compactification of the space of instantons on $\mathbb{C}^2/\mathbb{Z}_p$.

Torus fixed points for this compactification are labeled by the r-tuples of Young diagrams $\vec{\lambda^{\sigma}} = (\lambda_1^{\sigma_1}, \ldots, \lambda_r^{\sigma_r})$ colored in p colors, where $\sigma_j = 0, \ldots, p-1$ is a color of the angle of λ_j. Using this compactification the expression for the 4-point conformal block for algebras $\mathcal{A}(2,2)$ and $\mathcal{A}(2,4)$ was proposed in [7] and [8]. This result suggests the existence of the basis in the representation $\mathcal{A}(r,p)$ which is labeled by $\vec{\lambda^{\sigma}}$ such that matrix elements of the vertex operator have the form:

$$\begin{aligned}\langle J_{\vec{\lambda}^{\tilde{\sigma}}} | \Phi_\alpha | J_{\vec{\mu}^{\sigma}} \rangle = \prod_{S(\lambda_i,\mu_j)} (-b l_\lambda(s) + b^{-1} a_\mu(s) + b^{-1} - \alpha - P_i + \tilde{P}_j) \cdot \\ \cdot \prod_{S(\mu_j,\lambda_i)} (b l_\mu(t) + b - b^{-1} a_\lambda(t) + \alpha - P_i + \tilde{P}_j),\end{aligned} \tag{1.2}$$

where

$$s \in S(\lambda^{\tilde{\sigma}}, \mu^{\sigma}) \Longleftrightarrow s \in \lambda, \text{ and } l_\lambda(s) + a_\mu(s) + 1 + \sigma - \tilde{\sigma} \equiv 0 \bmod p,$$

a_λ and l_μ are the arm and leg length correspondingly, P_i and $\tilde{P}_j$ highest weights of the representations of $\mathcal{A}(r,p)$, b is related to the central charge of the algebra $\mathcal{A}(r,p)$, α is a parameter of the vertex operator.

In this paper we construct this basis for the algebras $\mathcal{A}(1,2)$ and $\mathcal{A}(2,2)$. It follows from the coset formula (1.1) that

$$\mathcal{A}(1,2) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_1, \qquad \mathcal{A}(2,2) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_2 \oplus \mathsf{NSR},$$

where $\mathcal{H}$ is a Heisenberg algebra, $\widehat{\mathfrak{sl}}(2)_k$ is affine algebra $\widehat{\mathfrak{sl}}(2)$ on the level k and NSR is the Neveu–Schwarz–Ramond algebra (the $\mathcal{N} = 1$ super–Virasoro algebra).

1.3. Denote by $\mathcal{E}_1(q,t)$ the quantum toroidal $\mathfrak{gl}(1)$ algebra [9] depending on two parameters q, t (another names are Ding-Iohara algebra, see [10], or elliptic Hall algebra, see [11]). We conjecture that in the limit $q, t \to -1$ of level r representations of $\mathcal{E}_1(q,t)$ we will have the representations of conformal algebra $\mathcal{A}(r,2)$. For example one can take the basis in the level r representations of $\mathcal{E}_1(q,t)$ introduced in [10] and obtain in the limit the basis in the representations of the algebra $\mathcal{A}(r,2)$. For the $r = 1$ case one can explicitly see the algebra $\mathcal{A}(1,2)$ in the limit of $\mathcal{E}_1(q,t)$. For $r = 2$ case we didn't prove the conjecture but support it by checks of three consequences: coincidence of characters, formulas for singular vectors and matrix elements of vertex operators.

One remark is in order. Denote by $\mathcal{E}_2(q,t)$ the quantum toroidal $\mathfrak{gl}(2)$ algebra. Then it is expected that in the conformal limit $q,t \to 1$ level r representations of $\mathcal{E}_2(q,t)$ we will also get conformal algebra $\mathcal{A}(r,2)$. Thus these algebras have different constructions as the limit of the deformed algebras. These conjectures have obvious generalizations for $p>2$.

1.4. Let us describe the results of our paper. For the $r=1$ the basis in representation of $\mathcal{E}_1(q,t)$ identified with the Macdonald polynomials $J_\lambda(q,t)$. Thus our basis identifies with the limit $q,t \to -1$ of the Macdonald polynomials. Such polynomials were introduced by Uglov in [12, 13], where he proved that these polynomials describe the eigenvectors of the spin generalization of the Calogero-Sutherland Model [14]. We call these polynomials *Uglov polynomials* as in [15]. It is worth to note that our idea of the limit $q,t \to -1$ of the algebra $\mathcal{E}_1(q,t)$ was based on the Uglov work. In addition to the Uglov results we see that the matrix elements of natural vertex operators (2.29),(2.30) has factorized form (1.2).

As was mentioned before in the limit of $\mathcal{E}_1$ we can see the algebra $\mathcal{A}(1,2)=\mathcal{H}\oplus\widehat{\mathfrak{sl}}(2)_1$. The last algebra comes in the principal (Lepowsky-Wilson) realization. Note that in the limit of toroidal algebra $\mathcal{E}_2(q,t)$ the algebra $\mathcal{A}(1,2)=\mathcal{H}\oplus\widehat{\mathfrak{sl}}(2)_1$ comes in homogenous (Frenkel-Kac) realization.

For the $r=2$ case we can take the limit of the basis from [10] in a similar way. The obtained basis $J_{\vec{\lambda}^\sigma}$ is labeled by pair of Young diagrams $\lambda_1^{\sigma_1}, \lambda_2^{\sigma_2}$ colored in two colors and has the following properties:

- If $\lambda_2=\varnothing$ then $J_{\lambda_1^{\sigma_1},\varnothing}$ is given in terms of Uglov polynomials after bosonization of algebra $\mathcal{A}(2,2)=\mathcal{H}\oplus\widehat{\mathfrak{sl}}(2)_2\oplus\mathsf{NSR}$ (see Proposition 3.1). Analogous facts for the algebra $\mathcal{A}(2,1)=\mathcal{H}\oplus\mathrm{Vir}$ were proved in [4].

  In the same way if $\lambda_1=\varnothing$ then $J_{\varnothing,\lambda_2^{\sigma_2}}$ is given in terms of Uglov polynomials but after *another* bosonization of $\mathcal{A}(2,2)$. In the special case $c_{\mathsf{NSR}}=\frac{3}{2}$ (in CFT notation $Q=b+\frac{1}{b}=0$ or in notation of Ω-deformation $\epsilon_1+\epsilon_2=0$) $J_{\vec{\lambda}^\sigma}$ is a product of two Uglov polynomials corresponding to diagrams $\lambda_1^{\sigma_1}$ and $\lambda_2^{\sigma_2}$.

- If $\lambda_2=\varnothing$, λ_1 is a rectangle $m\times n$, $m\equiv n (\mathrm{mod}\, 2)$ and highest weight of the representation $\mathcal{A}(2,2)$ take special value $P_{m,n}$ then the vectors $J_{\lambda_1^{\sigma_1},\varnothing}$ become the singular vectors for the NSR algebra. This singular vectors coincide with the Uglov polynomials after some nontrivial change of variables (see Proposition 4.1). This fact was checked up to level $9/2$. This result is a analogue of the description of the singular vectors for the Virasoro algebra in terms of the Jack polynomials [16, 17] [1].

- The matrix elements of the vertex operator $\Phi=\mathcal{V}_\alpha\cdot\Phi_\alpha^{\mathsf{NS}}$ have explicit factorized form (1.2), where $\mathcal{V}_\alpha$ is "rotated" Heisenberg vertex (2.29) and $\Phi_\alpha^{\mathsf{NS}}$ is the Neveu–Schwarz primary field. This fact was checked up to level 2 (see Proposition 3.2). The expression (1.2) also coincides with the limit of the matrix elements for vertex of the algebra $\mathcal{E}_1(q,t)$. Thus we expect that the limit of the last vertex is the operator $\Phi=\mathcal{V}_\alpha\cdot\Phi_\alpha^{\mathsf{NS}}$.

[1]see also [19] for the earlier results in this direction

Contents

2 $r = 1$ case

2.1 Algebra, homogeneous realization and characters

2.1.1. In this case we study the conformal field theory with the symmetry algebra $\mathcal{A}(1,2) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_1$. We denote the generators of the Heisenberg algebra $\mathcal{H}$ by w_n, where $n \in \mathbb{Z}$ and relations have the form:

$$[w_n, w_m] = 2n\delta_{n+m}.$$

The Lie algebra $\widehat{\mathfrak{sl}}(2) = \mathfrak{sl}(2) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ has generators $e_n = e \otimes t^n$, $f_n = f \otimes t^n$, $h_n = h \otimes t^n$ and the central generator K with relations:

$$[e_n, e_m] = [f_n, f_m] = 0, \quad [e_n, f_m] = h_{n+m} + n\delta_{n+m}K,$$
$$[h_n, e_m] = 2e_{n+m}, \quad [h_n, f_m] = -2f_{n+m}, \quad [h_n, h_m] = 2n\delta_{n+m}K.$$

Denote by $\mathcal{L}_{h,k}$ the integrable representation of $\widehat{\mathfrak{sl}}(2)$ generated by the highest vector v such that:

$$e_n v = 0, \text{ for } n \geq 0; \qquad f_n v = h_n v = 0, \text{ for } n > 0; \qquad h_0 v = hv, \ Kv = kv.$$

We denote by $\widehat{\mathfrak{sl}}(2)_k$ the quotient of the universal enveloping algebra, which acts on the integrable representations $\mathcal{L}_{h,k}$ (the representations of the level k). In this section we consider only level 1 representations.

Each integrable representation of $\widehat{\mathfrak{sl}}(2)$ has two natural grading. The first of them is a h_0 grading. Another grading is defined by Sugawara operator L_0 given by

$$L_0^{(\widehat{\mathfrak{sl}})} = \frac{1}{2(k+2)}\left(\sum_{k>0}(2e_{-k}f_k + 2f_{-k}e_k + h_{-k}h_k) + e_0f_0 + f_0e_0 + \frac{h_0^2}{2}\right).$$

This operator has very simple properties on the representations $\mathcal{L}_{h,k}$:

$$[L_0^{(\widehat{\mathfrak{sl}})}, e_{-n}] = ne_{-n}, \quad [L_0^{(\widehat{\mathfrak{sl}})}, h_{-n}] = nh_{-n}, \quad [L_0^{(\widehat{\mathfrak{sl}})}, f_{-n}] = nf_{-n}, \quad L_0^{(\widehat{\mathfrak{sl}})}v = \frac{h(h+2)}{4(k+2)}.$$

We refer to such grading as *homogeneous grading.* The character of the representation $\mathcal{L}_{h,k}$ defined by the formula

$$\chi_{\widehat{\mathfrak{sl}}}^{h,k} = \operatorname{Tr} q^{L_0}t^{h_0/2}\big|_{\mathcal{L}_{h,k}}.$$

2.1.2. The level 1 representations $\mathcal{L}_{0,1}$ and $\mathcal{L}_{1,1}$ have the simple construction due to Frenkel and Kac [20] in terms of one Heisenberg algebra h_n and the operator D . Denote by F_P the Fock representation of the Heisenberg algebra $\langle h_n\rangle$ with vacuum vector v_P:

$$h_nv_P = 0 \ \text{ for } n>0, \quad h_0v_P = Pv_P.$$

Denote by $D\colon \mathrm{F}_P \to \mathrm{F}_{P+1}$ operator defined by the commutation relations

$$[h_n, D] = 0 \ \text{ for } n \neq 0, \quad [h_0, D] = D.$$

Then $\widehat{\mathfrak{sl}}(2)_1$ representations $\mathcal{L}_{0,1}$ and $\mathcal{L}_{1,1}$ can be realized as the direct sums of the Fock modules, namely

$$\mathcal{L}_{0,1} = \bigoplus_{n\in\mathbb{Z}}\mathrm{F}_{2n}, \qquad \mathcal{L}_{1,1} = \bigoplus_{n\in\mathbb{Z}}\mathrm{F}_{2n+1}. \tag{2.1}$$

The generators e_i and f_i are defined in both representations by the formulae

$$e(z) = \sum_{n\in\mathbb{Z}} e_nz^{-n} = z^{h_0/2}D^2\exp\Big(2\sum_{n\in\mathbb{Z}_{>0}}\frac{h_{-n}}{2n}z^n\Big)\exp\Big(2\sum_{n\in\mathbb{Z}_{>0}}\frac{h_n}{-2n}z^{-n}\Big)z^{h_0/2},$$
$$f(z) = \sum_{n\in\mathbb{Z}} f_nz^{-n} = z^{h_0/2}D^2\exp\Big(-2\sum_{n\in\mathbb{Z}_{>0}}\frac{h_{-n}}{2n}z^n\Big)\exp\Big(-2\sum_{n\in\mathbb{Z}_{>0}}\frac{h_n}{-2n}z^{-n}\Big)z^{h_0/2}.$$

It is convenient to consider operators h_n as modes of bosonic field $\varphi(z)$:

$$\varphi(z) = \sum_{n\in\mathbb{Z}\setminus 0}\frac{h_n}{-2n}z^{-n} + \frac{h_0\log z}{2} + \widehat{Q}, \tag{2.2}$$

where the operator $\widehat{Q}$ is a conjugate to the operator $\widehat{P} = h_0$, i.e. defined by the relation: $[h_0, \widehat{Q}] = 1$. Therefore $D = \exp\widehat{Q}$. Using the field $\varphi(z)$ the free field realization of $\widehat{\mathfrak{sl}}(2)_1$ can be simply rewritten as:

$$\sum_{n\in\mathbb{Z}}e_nz^{-n} =:\exp\Big(2\varphi(z)\Big):, \quad \sum_{n\in\mathbb{Z}}f_nz^{-n} =:\exp\Big(-2\varphi(z)\Big):, \quad \sum_{n\in\mathbb{Z}}h_nz^{-n} = 2z\partial_z\varphi(z). \tag{2.3}$$

where $: \ldots :$ denotes the creation-annihilation normal ordering.

The formulas for characters of the representations $\mathcal{L}_{0,1}$ and $\mathcal{L}_{1,1}$ follow from the construction (2.1):

$$\chi^{0,1}_{\widehat{\mathfrak{sl}}} = \sum_{n\in\mathbb{Z}} t^n q^{n^2}\chi_B(q), \quad \chi^{1,1}_{\widehat{\mathfrak{sl}}} = \sum_{r\in\mathbb{Z}+\frac{1}{2}} t^r q^{r^2}\chi_B(q), \qquad \text{where } \chi_B(q) = \prod_{k\in\mathbb{Z}_{>0}} \frac{1}{1-q^k}. \quad (2.4)$$

2.1.3. We want to construct the special basis in the representation of algebra $\mathcal{A}(1,2) = \mathcal{H}\oplus\widehat{\mathfrak{sl}}(2)_1$. The basic vectors labeled by the torus fixed points on the moduli space $\bigsqcup_N \mathcal{M}(1,N)^{\mathbb{Z}_2}$. In this case the torus fixed points are labeled by two colors colored Young diagrams λ with angle colored in the color σ (see Appendix A for the notation on colored partitions). We will denote such basis by J_{λ^σ}. The color of the angle σ corresponds to the highest weight of representations $\mathcal{L}_{\sigma,1}$.

From the definition of the geometric action of operators e_{-1}, f_0, e_0, f_1 (see [25]) follows that combinatorial characteristics agree with algebraic grading as follows:

$$h_0(J_{\lambda^\sigma}) = (2d(\lambda^\sigma)+\sigma)J_{\lambda^\sigma}, \qquad L_0(J_{\lambda^\sigma}) = \frac{2|\lambda|+h_0}{4}J_{\lambda^\sigma},$$

where $d(\lambda^\sigma) = N_0(\lambda^\sigma) - N_1(\lambda^\sigma)$ is a difference between number of white and black boxes in diagram λ^σ, L_0 is a total degree with respect to the algebra $\mathcal{H}\oplus\widehat{\mathfrak{sl}}(2)_1$:

$$L_0 = L_0^{\widehat{\mathfrak{sl}}} + L_0^H = L_0^{\widehat{\mathfrak{sl}}} + \sum_{k\in\mathbb{Z}_{>0}} w_{-k}w_k.$$

The existence of such basis means that we can compute the character of representation of algebra $\mathcal{A}(1,2)$. Using the generating functions $\chi^{(1)}_{d,\sigma}$ introduced in Appendix A and formula (A.1) we get:

$$\begin{aligned}\chi^{(1)}_\sigma &:= \sum_{\lambda^\sigma} q^{\frac{|\lambda|+d(\lambda^\sigma)+\sigma/2}{2}} t^{d(\lambda^\sigma)+\sigma/2} = \sum_d (t\sqrt{q})^{d+\sigma/2}\chi^{(1)}_{d,\sigma}(q) = \\ &= \sum_d (t\sqrt{q})^{d+\sigma/2} q^{\frac{2d^2-(-1)^\sigma d}{2}}\chi_B(q)^2 = \sum_{n\in\mathbb{Z}+\frac{\sigma}{2}} t^n q^{n^2}\chi_B(q)^2.\end{aligned}$$

Due to formulas (2.4) the last expressions coincide with the characters of $\mathcal{H}\oplus\widehat{\mathfrak{sl}}(2)_1$ representations:

$$\mathcal{F}\otimes\mathcal{L}_{0,1} = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}\otimes \mathrm{F}_{2n}, \qquad \mathcal{F}\otimes\mathcal{L}_{1,1} = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}\otimes \mathrm{F}_{2n+1},$$

where $\mathcal{F}$ is a Fock representation of $\mathcal{H}$. Each vector in this representations can be obtained from the vacuum vector by action of w_n, h_n, D so it is natural to ask for the expression of J_{λ^σ} in terms of these generators.

2.2 The limit and principal realization

We will study the algebra $\mathcal{A}(1,2)$ using the limit of the quantum toroidal $\mathfrak{gl}_1$ algebra $\mathcal{E}_1(q,t)$ depending on two parameters q, t.

2.2.1. We will mainly follow [10] in this and next subsubsections.

The *quantum toroidal* $\mathfrak{gl}_1$ algebra is an associative algebra $U(q,t)$ generated by the E_i, F_i, $i \in \mathbb{Z}$, ψ^+_k, ψ^-_{-k}, $k \in \mathbb{Z}_{\geq 0}$ and central element C. The relations are written in terms of currents

$$E(z) = \sum_{n\in\mathbb{Z}} E_n z^{-n}, \quad F(z) = \sum_{n\in\mathbb{Z}} F_n z^{-n}, \quad \psi^{\pm}(z) = \sum_{\pm n\in\mathbb{Z}_{\geq 0}} \psi^{\pm}_n z^{-n}$$

and have the form:

$$\begin{aligned}
&\psi^{\pm}(z)\psi^{\pm}(w) = \psi^{\pm}(w)\psi^{\pm}(z), \qquad \psi^+(z)\psi^-(w) = \frac{g(C^2w/z)}{g(C^{-2}w/z)}\psi^-(w)\psi^+(z),\\
&\psi^+(z)E(w) = g(C^{-1}w/z)^{-1}E(w)\psi^+(z), \qquad \psi^-(z)E(w) = g(C^{-1}z/w)E(w)\psi^-(z),\\
&\psi^+(z)F(w) = g(Cw/z)F(w)\psi^+(z), \qquad \psi^-(z)F(w) = g(Cz/w)^{-1}F(w)\psi^-(z),\\
&[E(z),F(w)] = \frac{(1-q)(1-1/t)}{1-q/t}\big(\delta(C^{-2}z/w)\psi^+(Cw) - \delta(C^2z/w)\psi^-(C^{-1}w)\big),\\
&G^{-1}(z/w)E(z)E(w) = G(z/w)E(w)E(z), \qquad G(z/w)F(z)F(w) = G^{-1}(z/w)F(w)F(z),\\
&[E_0,[E_1,E_{-1}]] = [F_0,[F_1,F_{-1}]] = 0,
\end{aligned}$$

where

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n, \qquad g(z) := \frac{G^+(z)}{G^-(z)}, \quad G^{\pm}(z) := (1-q^{\pm 1}z)(1-t^{\mp 1}z)(1-q^{\mp 1}t^{\pm 1}z).$$

Note that elements ψ^+_0, ψ^-_0 are central. The representation V is said to be *of level* (r,l) if

$$Cv = (t/q)^{r/4}v, \qquad (\psi^-_0/\psi^+_0)v = (t/q)^l v$$

for any $v \in V$. In this paper we consider only the $l=0$ case so we will denote level just by r for simplicity.

It was proven in [21] that there exist the infinite system of commuting elements $I_1, I_2, \ldots$ in $\mathcal{E}_1$ (system of Integrals of motion). The first of them has the form $I_1 = E_0$.

2.2.2. The level 1 realization of $\mathcal{E}_1(q,t)$ is given in Fock representation on (deformed) Heisenberg algebra. Consider the algebra with generators a_n ($n \in \mathbb{Z}\setminus\{0\}$) and relations:

$$[a_n, a_m] = n\frac{1-q^{|n|}}{1-t^{|n|}}\delta_{n+m}. \tag{2.5}$$

Denote by $\mathcal{F}$ the Fock representation of this Heisenberg algebra generated by vacuum vector $|0\rangle$ such that

$$a_n|0\rangle = 0 \quad \text{for } n > 0.$$

It was proven in [21] that for any $u \in \mathbb{C}$ the following formulae give the level 1 represen-

tation of the algebra $\mathcal{E}_1$

$$
\begin{aligned}
&C \mapsto (t/q)^{1/4}\\
&E(z) \mapsto u \exp\Big(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} a_{-n} z^{n}\Big) \exp\Big(-\sum_{n=1}^{\infty} \frac{1-t^{n}}{n} a_{n} z^{-n}\Big),\\
&F(z) \mapsto u^{-1} \exp\Big(-\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} (t/q)^{n/2} a_{-n} z^{n}\Big) \exp\Big(\sum_{n=1}^{\infty} \frac{1-t^{n}}{n} (t/q)^{n/2} a_{n} z^{-n}\Big),\\
&\psi^{+}(z) \mapsto \exp\Big(-\sum_{n=1}^{\infty} \frac{1-t^{n}}{n} (1-t^{n}q^{-n})(t/q)^{-n/4} a_{n} z^{-n}\Big),\\
&\psi^{-}(z) \mapsto \exp\Big(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} (1-t^{n}q^{-n})(t/q)^{-n/4} a_{-n} z^{n}\Big).
\end{aligned}
\tag{2.6}
$$

We denote this representation by $\mathcal{F}_u$.

2.2.3. Consider the limit

$$
q = -e^{-\tau\epsilon_2},\ t = -e^{\tau\epsilon_1}, \qquad \tau \to 0,\ \epsilon_1 = b,\ \epsilon_2 = b^{-1}. \tag{2.7}
$$

The limit of the Heisenberg algebra (2.5) has relations:

$$
[a_n, a_m] = \begin{cases} -nb^{-2}\delta_{n+m,0}, & \text{if} \quad n \equiv 0 \bmod 2 \\ \quad n\delta_{n+m,0}, & \text{if} \quad n \equiv 1 \bmod 2 \end{cases}
$$

We see that the Heisenberg algebra falls into two pieces: even and odd part. For simplicity we make a replacement $a_{2n} \mapsto ib^{-1}a_{2n}$ and get the standard Heisenberg algebra

$$
[a_n, a_m] = \begin{cases} n\delta_{n+m,0}, & \text{if} \quad n \equiv 0 \bmod 2 \\ n\delta_{n+m,0}, & \text{if} \quad n \equiv 1 \bmod 2 \end{cases}
$$

The u (parameter of representation $\mathcal{F}_u$) in the limit reads: $u = (-1)^{\sigma} e^{\kappa\tau}$, where $\sigma \in \{0, 1\}$ is a discrete parameter and $\kappa \in \mathbb{C}$ is a continuous parameter. We denote the limit representation by $\mathcal{F}^{(\sigma)}(\kappa)$. The limit of generators in $\mathcal{F}^{(\sigma)}(\kappa)$ reads:

$$
\begin{aligned}
&E(z), F(z) \mapsto :(-1)^{\sigma} \exp\left(2\sum \frac{a_{2n+1}}{-(2n+1)} z^{-(2n+1)}\right): +O(\tau),\\
&\frac{\psi^{\pm}(z) - \psi^{\pm}(-z)}{2} \mapsto \mp\Big(2Q\sum\nolimits_{\pm n\in\mathbb{Z}_{>0}} a_{2n+1} z^{-(2n+1)}\Big)\tau + O(\tau^2),\\
&\frac{\psi^{\pm}(z) + \psi^{\pm}(-z)}{2} \mapsto 1 + \left(2Q^2\Big(\sum_{\pm n\in\mathbb{Z}_{>0}} a_{2n+1} z^{-(2n+1)}\Big)^2 - 2Q\epsilon_1 \sum_{\pm n\in\mathbb{Z}_{>0}} 2n a_{2n} z^{-2n}\right)\tau^2 + O(\tau^3).
\end{aligned}
\tag{2.8}
$$

The formula for the limit of $E(z)$ recovers the principal (Lepowsky–Wilson) realization of $\widehat{\mathfrak{sl}}(2)_1$ which we recall in the next paragraph.

2.2.4. One can introduce another set of generators in the algebra $\widehat{\mathfrak{sl}}(2)$:

$$
a_{2n+1} = f_{n+1} + e_n, \quad b_{2n+1} = f_{n+1} - e_n, \quad b_{2n} = h_n - \frac{1}{2}\delta_{n,0}K. \tag{2.9}
$$

This generators satisfy the relations

$$
\begin{aligned}
&[a_{2n+1}, a_{2m+1}] = (2n+1)\delta_{n+m+1}, \quad [a_{2n+1}, b_m] = 2b_{m+2n+1}, \quad [b_{2n}, b_{2m}] = 2n\delta_{m+n} \\
&[b_{2n+1}, b_{2m+1}] = -(2n+1)\delta_{n+m+1}, \quad [b_{2n+1}, b_{2m}] = 2a_{2m+2n+1}.
\end{aligned}
$$

It was proven by Lepowsky and Wilson [22] that integrable representations $\mathcal{L}_{\sigma,1}$ (where $\sigma = 0, 1$) can be realized as $\mathbb{C}[a_{-1}, a_{-3}, a_{-5}, \ldots]$ the Fock module over Heisenberg algebra a_{2n+1}. The action of the generators b_n on this representation is defined by the formula

$$
\sum_n b_n z^{-n} = b(z) = \frac{(-1)^{\sigma+1}}{2} \exp\left(2\sum_n \frac{a_{2n+1}}{-2n-1} z^{-2n-1}\right). \tag{2.10}
$$

Therefore, in the limit $\mathcal{E}_1(q,t)$ operators E_n and $\psi^{\pm}_{2n+1}$ give generators of $\widehat{\mathfrak{sl}}(2)$: b_n and a_{2n+1} correspondingly. The remaining set of generators $\psi^{\pm}_{2n}$ converges to the additional Heisenberg algebra w_n. If we denote the limit of representation $\mathcal{F}_u$, $u = (-1)^\sigma e^{\kappa\tau}$, $\tau \to 0$ by $\mathcal{F}^{(\sigma)}(\kappa)$ then we get:

$$
\mathcal{F}^{(\sigma)}(\kappa) = \mathcal{F} \otimes \mathcal{L}_{\sigma,1},
$$

where $\mathcal{F}$ is a Fock representation of $\mathcal{H}$. The parameter κ doesn't appear in the formulas for the generators but appear in the following formulas for the vertex operator.

We recapitulate part of this subsection as follows: there exist two bosonizations of algebra $\mathcal{A}(1,2)$. First of them based on homogeneous realization and has generators w_n, h_n, D. Second based on the principal realization and has generators a_n. Due to (2.3) and (2.10) the relation between these two constructions reads:

$$
w_n = a_{2n}, \quad z\partial\varphi(z^2) = \sum h_n z^{-2n} = \frac{1}{2} + \frac{(-1)^{\sigma+1}}{4}\Big(\exp(2\phi) + \exp(-2\phi)\Big) \tag{2.11}
$$

and

$$
a_{2n} = w_n, \quad z\partial\phi(z^2) = \sum a_{2n+1} z^{-2n-1} = z^{-1}\exp\Big(2\varphi(z^2)\Big) + z\exp\Big(-2\varphi(z^2)\Big),
$$

where we used field notations: $\varphi(z)$ introduced in (2.2) and

$$
\phi(z) = \sum_{n\in\mathbb{Z}} \frac{a_{2n+1}}{-2n-1} z^{-2n-1}. \tag{2.12}
$$

2.3 Basis

In this subsection we will construct the basis in the representations of the algebra $\mathcal{A}(1,2) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_1$ using the limit of the algebra $\mathcal{E}_1(q,t)$ described above.

2.3.1. Denote by ι the isomorphism between the Fock space $\mathcal{F}_u$ and the space of symmetric polynomials Λ:

$$
\iota : \mathcal{F}_u \xrightarrow{\sim} \Lambda, \;\; a_\lambda \mapsto p_\lambda, \qquad \text{where} \;\; p_\lambda = p_{\lambda_1} \cdot p_{\lambda_2} \cdot \ldots \cdot p_{\lambda_k}, \;\; p_n = \sum x_i^n.
$$

This isomorphism converts Shapovalov form on $\mathcal{F}_u$ to the Macdonald scalar product on Λ (B.3). The Macdonald polynomials $J_\lambda(q,t)$ (integral form, see Appendix B.1) form an orthogonal basis in Λ. Denote their preimages by the same letter J_λ. First examples

have the form:

$$\begin{aligned}
&J_\varnothing = |u\rangle, \qquad J_{(1)} = (1-t)a_{-1}|u\rangle,\\
&J_{(2)} = \left(\frac{1}{2}(1+q)(1-t)^2a_{-1}^2 + \frac{1}{2}(1-q)(1-t^2)a_{-2}\right)|u\rangle,\\
&J_{(1,1)} = \left(\frac{1}{2}(1-t)^2(1+t)(a_{-1}^2 - a_{-2})\right)|u\rangle.
\end{aligned}$$

The operator $I_1 = E_0$ under isomorphism ι is identified with Macdonald difference operator. This operator is diagonal on the basis J_λ

$$E_0|J_\lambda\rangle = u\varepsilon_\lambda|J_\lambda\rangle, \quad \varepsilon_\lambda := 1 + (t-1)\sum(q^{\lambda_i}-1)t^{-i}. \tag{2.13}$$

2.3.2. Macdonald polynomials $J_\lambda^{q,t}$ have a well defined limit when $q,t\to -1$ [12]. We denote this limit by *rank 2 Uglov polynomials* $J_\lambda^{(-1/b^2,2)}$. Many properties of this polynomials simply follow from the properties of Macdonald polynomials, we collect them in Appendix B.2. It is convenient to introduce $J_\lambda^{(2)}$ by:

$$J_\lambda^{(2)} = \lim_{\tau\to\infty}\left(\frac{(-1)^{n(\lambda)}}{\tau^{|\lambda^\diamond|}\,2^{|\lambda|-|\lambda^\diamond|}}\,J_\lambda(q,t)\right). \tag{2.14}$$

First examples have the form:

$$\begin{aligned}
&J_\varnothing^{(2)} = |\kappa\rangle^\sigma, \qquad J_{(1)}^{(2)} = a_{-1}|\kappa\rangle^\sigma, \qquad J_{(2)}^{(2)} = (b^{-1}a_{-1}^2 - ia_{-2})|\kappa\rangle^\sigma,\\
&J_{(1,1)}^{(2)} = (ba_{-1}^2 - ia_{-2})|\kappa\rangle^\sigma, \qquad J_{(3)}^{(2)} = (\frac{1}{3b}a_{-1}^3 - ia_{-2}a_{-1} + \frac{2}{3b}a_{-3})|\kappa\rangle^\sigma,\\
&J_{(2,1)}^{(2)} = -\frac{1}{3}(a_{-1}^3 - a_{-3})|\kappa\rangle^\sigma, \qquad J_{(1,1,1)}^{(2)} = (\frac{b}{3}a_{-1}^3 - ia_{-2}a_{-1} + \frac{2b}{3}a_{-3})|\kappa\rangle^\sigma,\\
&J_{(2,2)}^{(2)} = (-iQa_{-4} + \frac{4}{3}a_{-3}a_{-1} - a_{-2}^2 - \frac{1}{3}a_{-1}^4)|\kappa\rangle^\sigma.
\end{aligned} \tag{2.15}$$

Here and below $|\kappa\rangle^\sigma$ denotes the vacuum vector in the representation $\mathcal{F}^{(\sigma)}(\kappa)$. We assign to each vector $J_\lambda^{(2)} \in \mathcal{F}^{(\sigma)}(\kappa)$ the Young diagram λ colored in two colors with the angle of color σ (see Appendix A for the definition and notation $d(\lambda)$, $\lambda^\diamond$).

From the orthogonality of Macdonald polynomials follows that $J_\lambda^{(2)}|\kappa\rangle^\sigma$ are orthogonal with norms given by (see (B.6)):

$$\langle J_\lambda^{(2)}|J_\mu^{(2)}\rangle = \delta_{\lambda,\mu}(-1)^{|\lambda^\diamond|}\prod_{s\in\lambda^\diamond}\left(bl_\lambda(s) + b - \frac{a_\lambda(s)}{b}\right)\left(-bl_\lambda(s) + \frac{a_\lambda(s)+1}{b}\right), \tag{2.16}$$

where $\lambda^\diamond = \{s\in\lambda \mid a_\lambda(s) + l_\lambda(s) + 1 \equiv 0 \mod 2\}$.

2.3.3. In the limit of integrable system in the algebra $\mathcal{E}_1(q,t)$ one can find the system of Integrals of Motion which act diagonally on the basis $J_\lambda^{(2)}$. For example from formula (2.13) follows that

$$e_0 \to 1 - 2h_0, \qquad h_0(J_{\lambda^\sigma}^{(2)}) = (2d(\lambda) + q)\,J_{\lambda^\sigma}^{(2)}. \tag{2.17}$$

We add superscript σ (color of the angle of λ^σ or the highest weight of $\mathcal{L}_{\sigma,1}$) in notation where formulas depend on it. The formula for action of h_0 (2.17) is equivalent to the fact

that the vectors

$$J^{(2)}_{\lambda^0},\ d(\lambda)=d \quad \text{form a basis in} \quad \mathcal{F}\otimes \mathrm{F}_{2d}, \tag{2.18}$$

$$J^{(2)}_{\lambda^1},\ d(\lambda)=d \quad \text{form a basis in} \quad \mathcal{F}\otimes \mathrm{F}_{2d+1}. \tag{2.19}$$

The vectors $J^{(2)}_{\lambda^\sigma}$ with $|\lambda|$ minimal for given $d(\lambda)$ are the highest vectors in $\mathcal{F}\otimes \mathrm{F}_d$. Such λ has triangular form and called 2-core (see Appendix A).

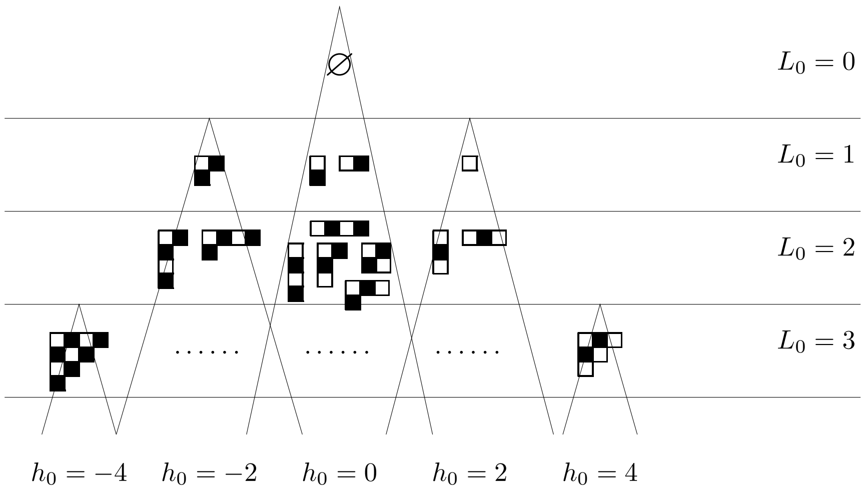

Figure 1: The basis in $\mathcal{F}^{(0)}(\kappa)$. Colored diagram λ represents a vector $J^{(2)}_{\lambda^0}$. The interior of each angle corresponds to the representation $\mathcal{F}\otimes \mathrm{F}_{2d}\subset \mathcal{F}^{(0)}(\kappa)$.

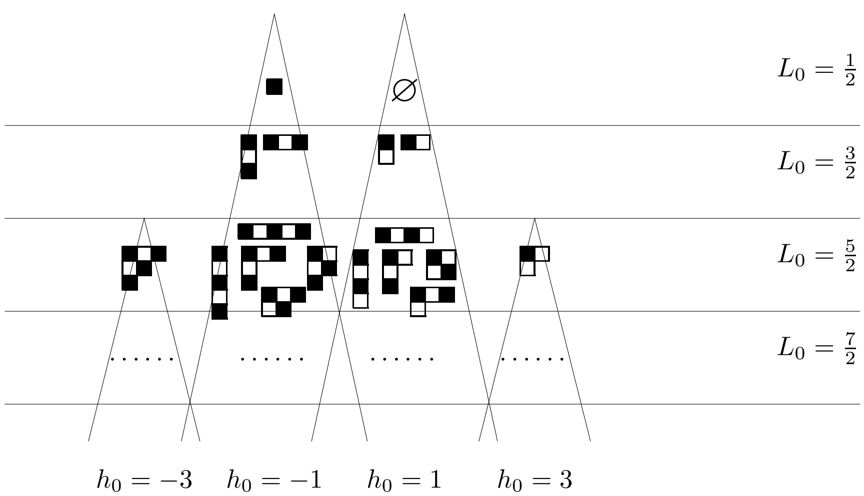

Figure 2: The basis in $\mathcal{F}^{(1)}(\kappa)$.

2.3.4. It is convenient to rewrite the basis $J^{(2)}_{\lambda^\sigma}$ in terms of homogeneous bosonization (with generators w_n, h_n, D). First note that D commute with h_n, w_n for $n\neq 0$. Then action of D translates the highest vector of $\mathcal{F}\otimes \mathrm{F}_d$ to the highest vector of $\mathcal{F}\otimes \mathrm{F}_{d+1}$. From (2.16) follows that norms $\langle J^{(2)}_{\lambda^\sigma}|J^{(2)}_{\lambda^\sigma}\rangle = 1$ if λ is a 2-core. Hence all such vectors can

be obtained by the action of D to the vacuum $|\kappa\rangle^\sigma$:

$$J^{(2)}_{\varnothing^0} = |\kappa\rangle^0, \quad J^{(2)}_{(1)^0} = D^2|\kappa\rangle^0, \quad J^{(2)}_{(2,1)^0} = D^{-2}|\kappa\rangle^0, \quad J^{(2)}_{(3,2,1)^0} = D^4|\kappa\rangle^0, \dots$$
$$J^{(2)}_{\varnothing^1} = |\kappa\rangle^1, \quad J^{(2)}_{(1)^1} = D^{-2}|\kappa\rangle^1, \quad J^{(2)}_{(2,1)^1} = D^2|\kappa\rangle^1, \quad J^{(2)}_{(3,2,1)^1} = D^{-4}|\kappa\rangle^1, \dots$$

Any other $J^{(2)}_{\lambda^\sigma}$ can be obtained from $J^{(2)}_{\widetilde{\lambda}^\sigma}$ by action of generators of the two Heisenberg algebras $h_n \in \widehat{\mathfrak{sl}}(2)$ and $w_n \in \mathcal{H}$, where $\widetilde{\lambda}$ is 2–core of partition λ. The first examples in the space $\mathcal{F} \otimes \mathrm{F}_0$ are:

$$\begin{aligned}
&J^{(2)}_{\varnothing^0} = |\kappa\rangle^0 \quad J^{(2)}_{(2)^0} = -(iw_{-1} + b^{-1}h_{-1})|\kappa\rangle^0, \quad J^{(2)}_{(1,1)^0} = -(iw_{-1} + bh_{-1})|\kappa\rangle^0,\\
&J^{(2)}_{(1,1,1,1)^0} = (-2ibw_{-2} - w_{-1}^2 + 2ibw_{-1}h_{-1} + bh_{-1}^2 - 2bh_{-2})|\kappa\rangle^0,\\
&J^{(2)}_{(2,1,1)^0} = \left(-2ibw_{-2} - w_{-1}^2 + i(b^{-1} - b)w_{-1}h_{-1} + b^2h_{-1}^2 - (1 - b^2)h_{-2}\right)|\kappa\rangle^0,\\
&J^{(2)}_{(2,2)^0} = \left(-i(b + b^{-1})w_{-2} - w_{-1}^2 + h_{-1}^2\right)|\kappa\rangle^0.
\end{aligned}$$

The first examples in the space $\mathcal{F} \otimes \mathrm{F}_2$ are:

$$\begin{aligned}
&J^{(2)}_{(1)^0} = D^2|\kappa\rangle^0, \quad J^{(2)}_{(3)^0} = -(iw_{-1} - b^{-1}h_{-1})D^2|\kappa\rangle^0, \quad J^{(2)}_{(1,1,1)^0} = -(iw_{-1} - bh_{-1})D^2|\kappa\rangle^0,\\
&J^{(2)}_{(1,1,1,1,1)^0} = (-2ibw_{-2} - w_{-1}^2 - 2ibw_{-1}h_{-1} + bh_{-1}^2 + 2bh_{-2})D^2|\kappa\rangle^0,\\
&J^{(2)}_{(2,2,1)^0} = \left(i(b - b^{-1})w_{-2} - w_{-1}^2 - 2ibw_{-1}h_{-1} + b^2h_{-1}^2 + (1 - b^2)h_{-2}\right) D^2|\kappa\rangle^0,\\
&J^{(2)}_{(3,1,1)^0} = \left(-i(b + b^{-1})w_{-1}h_{-1} - w_{-1}^2 + h_{-1}^2\right) D^2|\kappa\rangle^0.
\end{aligned}$$

Note that in the examples above the coefficients of $J^{(2)}_{\lambda}$ in terms of h_n, w_n become simpler.

2.4 Vertex operator

In this subsection we study the vertex operators for the deformed algebra $\mathcal{E}_1(q,t)$ and then pass to the limit $q, t \to -1$. Remarkably this limit operator is a natural vertex operator for the algebra $\mathcal{A}(1,2) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_1$.

2.4.1. Awata et al [10] introduced the vertex operators for the algebra $\mathcal{E}_1(q,t)$. As usual these operator can be defined by commutation relations with the generators of the algebra. The level 1 vertex operator $\Phi(z)\colon \mathcal{F}_u \to \mathcal{F}_v$ defining relations read:

$$\begin{aligned}
&(1 - v\frac{w}{z})E(z)\Phi(w) = (1 - q^{-1}tv\frac{w}{z})\Phi(w)E(z),\\
&(1 - (t/q)^{3/2}u\frac{w}{z})F(z)\Phi(w) = (1 - (t/q)^{1/2}uw/z)\Phi(w)F(z),\\
&(1 - (t/q)^{-1/4}vw/z)(1 - (t/q)^{7/4}u\frac{w}{z})\psi^+(z)\Phi(w)\\
&\qquad = (1 - (t/q)^{3/4}v\frac{w}{z})(1 - (t/q)^{3/4}u\frac{w}{z})\Phi(w)\psi^+(z),\\
&(1 - (t/q)^{1/4}v\frac{w}{z})(1 - (t/q)^{5/4}u\frac{w}{z})\psi^-(z)\Phi(w)\\
&\qquad = (1 - (t/q)^{5/4}v\frac{w}{z})(1 - (t/q)^{1/4}u\frac{w}{z})\Phi(w)\psi^-(z).
\end{aligned} \tag{2.20}$$

In the level 1 case authors of [10] provided an explicit exponential form for this operator:

$$\Phi(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{v^n - (t/q)^n u^n}{1 - q^n} \cdot \frac{a_{-n}z^n}{n}\right) \exp\left(\sum_{n=1}^{\infty} \frac{v^{-n} - u^{-n}}{1 - q^{-n}} \cdot \frac{a_n z^{-n}}{n}\right). \tag{2.21}$$

The following proposition was stated in [10].

Proposition 2.1. *The matrix elements of the operator $\Phi(z)$ in basis J_λ have factorized (Nekrasov) form*

$$\langle J_\lambda|\Phi(z)|J_\mu\rangle = N_{\lambda,\mu}\left(\frac{qv}{tu}\right)\cdot\left(\frac{tu}{q}\right)^{|\lambda|}\left(-\frac{v}{q}\right)^{-|\mu|} t^{n(\lambda)}q^{n(\mu')}z^{|\lambda|-|\mu|}, \tag{2.22}$$

where $n(\lambda)$ defined in (B.1) *and*

$$N_{\lambda,\mu}(u) = \prod_{s\in\lambda}(1-uq^{-a_\mu(s)-1}t^{-l_\lambda(s)})\cdot\prod_{t\in\mu}(1-uq^{a_\lambda(t)}t^{l_\mu(t)+1}). \tag{2.23}$$

arm $a_\lambda(s)$ and leg $l_\lambda(s)$ lengths defined in (A.2).

2.4.2. The vertex operator (2.21) depend on parameters u and v. We parameterize them by $u = (-1)^\sigma e^{\kappa\tau}$ and $v = (-1)^{\tilde\sigma}e^{\tilde\kappa\tau}$ and then pass to the limit $\tau\to 0$. We have two natural cases: in the first case $\sigma=\tilde\sigma$, in the second $\sigma\neq\tilde\sigma$.

$\Phi_0(z)\colon \mathcal{F}^{(0)}(\kappa)\to\mathcal{F}^{(0)}(\tilde\kappa),\ \mathcal{F}^{(1)}(\kappa)\to\mathcal{F}^{(1)}(\tilde\kappa)\quad \Phi_1(z)\colon \mathcal{F}^{(0)}(\kappa)\to\mathcal{F}^{(1)}(\tilde\kappa),\ \mathcal{F}^{(1)}(\kappa)\to\mathcal{F}^{(0)}(\tilde\kappa)$

$$\Phi_0(z) = \exp\left(i(\alpha-Q)\sum_{n=1}^\infty\frac{a_{-2n}z^{2n}}{-2n}\right)\exp\left(i\alpha\sum_{n=1}^\infty\frac{a_{2n}z^{-2n}}{2n}\right) = \mathcal{V}_\alpha, \tag{2.24}$$

$$\begin{aligned}\Phi_1(z) =& \exp\left(i(\alpha-Q)\sum_{n=1}^\infty\frac{a_{-2n}z^{2n}}{-2n}\right)\exp\left(i\alpha\sum_{n=1}^\infty\frac{a_{2n}z^{-2n}}{2n}\right)\cdot\exp\left(\sum_n\frac{a_{2n+1}}{-2n-1}z^{-2n-1}\right) =\\ =&\mathcal{V}_\alpha\cdot\exp(\phi) = \mathcal{V}_\alpha\cdot\mathcal{W},\end{aligned} \tag{2.25}$$

where $\alpha = \dfrac{\kappa-\tilde\kappa}{\sqrt{\epsilon_1\epsilon_2}}$. Note that $\mathcal{V}_\alpha$ is the standard rotated vertex operator for the Heisenberg algebra $\mathcal{H}$ (see [23]). The operator $\mathcal{W}$ is the vertex operator which acts on $\widehat{\mathfrak{sl}}(2)_1$ part. This vertex operator permute $\mathcal{L}_{0,1}$ and $\mathcal{L}_{1,1}$.

Note that formula (2.25) is similar to eq. (1) in [24].

2.4.3. Recall that in the limit $\tau\to 0$ the basis vectors $J_\lambda(q,t)$ tend to 0 as $\tau^{|\lambda^\diamond|}$. For the limit of matrix element we can have three possibilities. In the first, the matrix elements $N_{\lambda,\mu}(qv/tu)$ (see (2.22)) tend to 0 as $\tau^{|\lambda^\diamond|+|\mu^\diamond|}$. In this case the limit of matrix element is nonzero. If $N_{\lambda,\mu}(qv/tu)$ tends to 0 faster $\tau^{|\lambda^\diamond|+|\mu^\diamond|}$ then the limit of matrix element is zero. At last, if $N_{\lambda,\mu}(qv/tu)$ tends to 0 slowly then $\tau^{|\lambda^\diamond|+|\mu^\diamond|}$ we have pole in the matrix element.

By combinatorial consideration one can see that the only first two options exist in our case. The precise statements given in the following propositions. We will use:

$$N^{(2)}_{\lambda^{\tilde\sigma},\mu^\sigma}(\alpha) = \prod_{S(\lambda^{\tilde\sigma},\mu^\sigma)}(-bl_\lambda(s)+\frac{a_\mu(s)+1}{b}-\alpha)\cdot\prod_{S(\mu^\sigma,\lambda^{\tilde\sigma})}(b(l_\mu(t)+1)-\frac{a_\lambda(s)}{b}+\alpha), \tag{2.26}$$

where

$$s\in S(\lambda^{\tilde\sigma},\mu^\sigma)\iff s\in\lambda,\ \text{and}\ l_\lambda(s)+a_\mu(s)+1+\sigma-\tilde\sigma\equiv 0(\mathrm{mod}\,2). \tag{2.27}$$

Proposition 2.2. *Let $\sigma=\tilde\sigma$; $\Phi_0(z)\colon\mathcal{F}^\sigma(\kappa_1)\to\mathcal{F}^{\tilde\sigma}(\kappa_2)$. Then the limit*

$$\lim_{\tau\to 0}\left(\tau^{-|\lambda^\diamond|-|\mu^\diamond|}N_{\lambda,\mu}(qv/tu)\right)\neq 0\quad \textit{iff}\quad d(\lambda)^\sigma = d(\mu^\sigma).$$

In this case matrix element equals:

$$\langle J^{(2)}_{\lambda}|\Phi_0(z)|J^{(2)}_{\mu}\rangle = (-1)^{|\mu^{\diamondsuit}|}z^{|\lambda^{\diamondsuit}|-|\mu^{\diamondsuit}|}N^{(2)}_{\lambda^{\sigma},\mu^{\tilde{\sigma}}}(\alpha).$$

Proposition 2.3. *Let* $\sigma-\tilde{\sigma}\equiv 1 \bmod 2$*;* $\Phi_1(z)\colon \mathcal{F}^{\sigma}(\kappa_1)\to\mathcal{F}^{\tilde{\sigma}}(\kappa_2)$*. Then the limit*

$$\lim_{\tau\to 0}\Big(\tau^{-|\lambda^{\diamondsuit}|-|\mu^{\diamondsuit}|}N_{\lambda,\mu}(qv/tu)\Big)\neq 0 \quad \textit{iff} \quad 2d(\lambda^{\sigma})+\tilde{\sigma}=2d(\mu^{\tilde{\sigma}})+\sigma\pm 1.$$

In this case matrix element equals:

$$\langle J^{(2)}_{\lambda}|\Phi_1(z)|J^{(2)}_{\mu}\rangle = (-1)^{|\mu-\mu^{\diamondsuit}|}z^{|\lambda^{\diamondsuit}|-|\mu^{\diamondsuit}|}N^{(2)}_{\lambda^{\sigma},\mu^{\tilde{\sigma}}}(\alpha). \tag{2.28}$$

The combinatorial conditions in propositions 2.2 and 2.3 have a clear algebraic meaning. Namely the condition $d(\lambda)^{\sigma}=d(\mu^{\sigma})$ means that h_0 gradings on $J^{(2)}_{\lambda^{\sigma}}$ and $J^{(2)}_{\mu^{\sigma}}$ coincide. The condition $2d(\lambda^{\tilde{\sigma}})+\tilde{\sigma}=2d(\mu^{\sigma})+\sigma\pm 1$ means that h_0 gradings on $J^{(2)}_{\lambda^{\sigma}}$ and $J^{(2)}_{\mu^{\sigma}}$ differs on 1. In particular, we see that operator $\mathcal{W}$ shifts the h_0 grading by 1.

2.4.4. The vertex operators Φ_0,Φ_1 ((2.24), (2.25)) can be also rewritten in terms of h_n and w_n. It is evident that Φ_0 can be rewritten:

$$\Phi_0(z)=\mathcal{V}_{\alpha}(z)=\exp\Big(i(\alpha-Q)\sum_{n=1}^{\infty}\frac{w_{-n}z^{2n}}{-2n}\Big)\exp\Big(i\alpha\sum_{n=1}^{\infty}\frac{w_n z^{-2n}}{2n}\Big). \tag{2.29}$$

In order to rewrite Φ_1 we need to rewrite the $\widehat{\mathfrak{sl}}(2)$ nontrivial part $\mathcal{W}$ (2.25). Introduce two operators:

$$\begin{aligned}\mathcal{W}_+(z)&=z^{h_0/4}D\exp\Big(\sum_{n\in\mathbb{Z}_{>0}}\frac{h_{-n}}{2n}z^n\Big)\exp\Big(\sum_{n\in\mathbb{Z}_{>0}}\frac{h_n}{-2n}z^{-n}\Big)z^{h_0/4}=:\exp\Big(\varphi(z)\Big):,\\ \mathcal{W}_-(z)&=z^{-h_0/4}D^{-1}\exp\Big(\sum_{n\in\mathbb{Z}_{>0}}\frac{h_{-n}}{-2n}z^n\Big)\exp\Big(\sum_{n\in\mathbb{Z}_{>0}}\frac{h_n}{2n}z^{-n}\Big)z^{-h_0/4}=:\exp\Big(-\varphi(z)\Big):,\end{aligned} \tag{2.30}$$

where $\varphi(z)$ defined in (2.2). The operator $\mathcal{W}_+$ increases the h_0 grading by 1 and the operator $\mathcal{W}_-$ decreases the h_0 grading by 1. Under the operator-state correspondence these operators correspond to vectors $|\kappa\rangle^1$ and $f_{-1}|\kappa\rangle^1$ (equivalently $J^{(2)}_{\varnothing^1}$ and $J^{(2)}_{(1)^1}$) on the top of representation $\mathcal{L}_{1,1}$, see picture 2).

Proposition 2.4. *The operator* $\mathcal{W}$ *has the form*

$$\mathcal{W}=z^{1/2}\mathcal{W}_-(z^2)+z^{-1/2}\mathcal{W}_+(z^2) \tag{2.31}$$

The rescaling $z\to z^2$ and factors $z^{\pm 1/2}$ are just the transformation from the homogeneous grading to the principal grading. If we consider $\mathcal{W}\colon \mathcal{L}_{0,1}\to\mathcal{L}_{1,1}$ then $z^{-1/2}\mathcal{W}_+(z^2)$ consists of terms of even degrees of z and $z^{1/2}\mathcal{W}_-(z^2)$ consists of terms of odd degrees of z. If we consider $\mathcal{W}\colon \mathcal{L}_{1,1}\to\mathcal{L}_{0,1}$ then $z^{-1/2}\mathcal{W}_+(z^2)$ consists of terms of odd degrees of z and $z^{1/2}\mathcal{W}_-(z^2)$ consist of terms of even degrees of z.

Therefore the operators $\mathcal{V}_{\alpha}(z)\mathcal{W}_+(z)$ and $\mathcal{V}_{\alpha}(z)\mathcal{W}_+(z)$ have factorized matrix elements given by formula (2.28) or 0. Recall that the matrix elements of $\mathcal{V}_{\alpha}(z)\mathcal{W}_+(z)$ are nonzero if h_0 grading increases by 1 and the matrix elements of $\mathcal{V}_{\alpha}(z)\mathcal{W}_-(z)$ are nonzero if h_0 grading decreases by 1.

3 $r = 2$ case

3.1 Algebras, representations, characters

3.1.1. As was already mentioned in the Introduction we want to construct the special (geometric) basis in the representations of the algebra $\mathcal{A}(2,2) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_2 \oplus \mathsf{NSR}$. In this paper we consider only the Neveu–Schwarz sector of this algebra i.e. the NSR algebra is generated by L_n, G_r, $n \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}$ subject of relations:

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c_{\mathsf{NSR}}}{8}(n^3 - n)\delta_{n+m}, \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{1}{2}c_{\mathsf{NSR}}(r^2 - \frac{1}{4})\delta_{r+s}, \\
[L_n, G_r] &= \left(\frac{1}{2}n - r\right) G_{n+r},
\end{aligned} \tag{3.1}$$

where the central charge c_{NSR} is parameterized as follows

$$c_{\mathsf{NSR}} = 1 + 2Q^2, \qquad Q = b + \frac{1}{b}.$$

We will consider the representation of the algebra $\mathcal{A} = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_2 \oplus \mathsf{NSR}$ which is a tensor product of representations of the algebras $\mathcal{H}$, $\widehat{\mathfrak{sl}}(2)_2$ and NSR. For the Heisenberg algebra $\mathcal{H}$ we take standard Fock module $\mathcal{F}$ with the character χ_B, see (2.4). For the NSR we take the Verma module with the highest weight parameterized by the parameter P, namely the highest weight vector is denoted by $|P\rangle$ and defined by properties

$$L_n|P\rangle = G_r|P\rangle = 0 \quad \text{for} \quad n, r > 0, \qquad L_0|P\rangle = \Delta_{\mathsf{NS}}(P)|P\rangle,$$

where $\Delta_{\mathsf{NS}}(P) = \frac{1}{2}(Q^2/4 - P^2)$. We denote this representation as $\mathsf{V}_{\Delta_{\mathsf{NS}}}$ or simply V_Δ. The character of V_Δ equals

$$\chi_{\mathsf{NSR}}(q) = \operatorname{Tr} q^{L_0}\big|_{\mathsf{V}_\Delta} = q^\Delta \chi_B(q)\chi_F(q), \quad \text{where} \quad \chi_F(q) = \prod_{r+\frac{1}{2} \in \mathbb{Z}_{>0}} (1 + q^r). \tag{3.2}$$

The $\widehat{\mathfrak{sl}}(2)$ algebra has three integrable representations of level 2: $\mathcal{L}_{0,2}$, $\mathcal{L}_{1,2}$ and $\mathcal{L}_{2,2}$. The representation $\mathcal{L}_{0,2} \oplus \mathcal{L}_{2,2}$ is called the Neveu–Schwarz representation of $\widehat{\mathfrak{sl}}(2)_2$. The character of this representation equals [26][eq 4.2]

$$\chi_{\widehat{\mathfrak{sl}}}^{(0,2)}(q,t) + \chi_{\widehat{\mathfrak{sl}}}^{(2,2)}(q,t) = \prod_{r+\frac{1}{2} \in \mathbb{Z}_{>0}} (1 + t^{-1}q^r)(1 + q^r)(1 + tq^r) = \sum_{n \in \mathbb{Z}} t^n q^{n^2/2} \chi_B(q)\chi_F(q). \tag{3.3}$$

Note that the remaining representation $\mathcal{L}_{1,2}$ is called the Ramond representation of $\widehat{\mathfrak{sl}}(2)_2$.

3.1.2. We want to construct basis in the representation

$$\mathcal{F} \otimes (\mathcal{L}_{0,2} \oplus \mathcal{L}_{2,2}) \otimes \mathsf{V}_\Delta \tag{3.4}$$

of the algebra $\mathcal{A}(2,2)$ with the properties from the Introduction. The basic vectors labeled by torus fixed points on the moduli space $\bigsqcup_N \mathcal{M}(2,N)^{\mathbb{Z}_2}$. In this case torus fixed points are labeled by pairs of Young diagrams λ_1, λ_2 colored in two colors with angles colored in

the same color σ. We will denote such a basis by $J_{\lambda_1^\sigma,\lambda_2^\sigma}$ or $J_{\vec{\lambda}^\sigma}$ for short. As in $r=1$ case such basis respects grading:

$$h_0(J_{\vec{\lambda}^\sigma}) = (2d(\lambda_1^\sigma)+2d(\lambda_2^\sigma)+2\sigma)J_{\vec{\lambda}^\sigma}, \qquad L_0(J_{\vec{\lambda}^\sigma}) = (\frac{2|\lambda|+h_0}{4}+\Delta)J_{\vec{\lambda}^\sigma}. \tag{3.5}$$

The existence of this basis means that we can compute the character of representation of algebra $\mathcal{A}(2,2)$. Using the formula (A.1) we get:

$$\chi^{(2)}(q,t) := \sum_{\sigma=0,1}\sum_{\lambda_1^\sigma,\lambda_2^\sigma} q^{\frac{|\vec{\lambda}|+d(\vec{\lambda})+\sigma}{2}+\Delta}t^{d(\vec{\lambda})+\sigma} = q^\Delta \sum_{\sigma,d_1,d_2}(t\sqrt{q})^{d_1+d_2+\sigma}\chi^{(1)}_{\sigma,d_1}(q)\chi^{(1)}_{\sigma,d_2}(q) =$$
$$=q^\Delta\chi_B(q)^4\sum_{d_1,d_2}\left((t\sqrt{q})^{d_1+d_2}q^{\frac{2d_1^2+2d_2^2-d_1-d_2}{2}}+(t\sqrt{q})^{d_1+d_2+1}q^{\frac{2d_1^2+2d_2^2+d_1+d_2}{2}}\right)=$$
$$=q^\Delta\chi_B(q)^4\sum_d(t\sqrt{q})^d\sum_{d_1}\left(q^{\frac{(2d_1-d)^2+d^2-d}{2}}+q^{\frac{(2d_1-d+1)^2+d^2-d}{2}}\right)=q^\Delta\chi_B(q)^4\sum_{d,k}q^{\frac{d^2+k^2}{2}}t^d.$$

Using the Jacobi triple product identity this expression can be rewritten as

$$\chi^{(2)}(q,t) = q^\Delta\chi_B(q)^4\sum_{d,k}q^{\frac{d^2+k^2}{2}}t^d = q^\Delta\chi_B(q)^2\prod_{r+\frac{1}{2}\in\mathbb{Z}_{>0}}(1+t^{-1}q^r)^2(1+q^r)(1+tq^r) =$$
$$=\chi_B(q)\cdot\left(\chi^{(0,2)}_{\widehat{\mathfrak{sl}}}(q,t)+\chi^{(2,2)}_{\widehat{\mathfrak{sl}}}(q,t)\right)\cdot q^\Delta\chi_F(q)\chi_B(q) = \chi_{\mathcal{H}\oplus\widehat{\mathfrak{sl}}(2)_2\oplus\mathsf{NSR}},$$

where we used the formulas (3.2) and (3.3).

3.1.3. The generators of $\mathcal{H}$ are denoted by w_n as in $r=1$ case but we rescale commutation relations to:

$$[w_n,w_m] = 4n\delta_{n+m}.$$

So the full set of generators of algebra $\mathcal{A}(2,2)$ consist of $w_n; e_n, f_n, h_n; L_n, G_r$.

We will use the free field realization of the NSR and $\widehat{\mathfrak{sl}}(2)_2$ algebras. The free field realization means that we consider the Fock representation of Heisenberg algebra and Majorana fermion algebra and define the action of the mentioned before algebras on this representation. This method is very useful since Heisenberg and Majorana fermion algebra are much simpler.

Recall the Fateev–Zamolodchikov [27] realization of algebra $\widehat{\mathfrak{sl}}(2)_2$. Introduce generators h_n, D, χ_r:

$$[h_n,h_m] = 4n\delta_{n+m,0}, \quad \{\chi_r,\chi_s\} = \delta_{r+s,0}, \quad [h_n,\chi_r]=0.$$

Clearly h_n generate the Heisenberg algebra $\mathcal{H}$, and χ_r generate the Fermion algebra F. Denote by F_P the Fock representation of algebra $\mathcal{H}\oplus\mathsf{F}$ with vacuum vector v_P:

$$h_nv_P=0,\ \chi_rv_p=0 \quad \text{for } n,r>0; \qquad h_0v_P=Pv_P.$$

As in $r=1$ case we introduce the operator $D\colon \mathrm{F}_P\to\mathrm{F}_{P+1}$ defined by commutation relations

$$[D,h_n]=0 \text{ for } n\neq 0, \quad [h_0,D]=D, \quad [\chi_r,D]=0.$$

Then $\widehat{\mathfrak{sl}}(2)_2$ representations $\mathcal{L}_{0,2}\oplus\mathcal{L}_{2,2}$ can be realized as the direct sum:

$$\mathcal{L}_{0,2}\oplus\mathcal{L}_{2,2} = \bigoplus_{d\in\mathbb{Z}}\mathrm{F}_{2d}. \tag{3.6}$$

The operators e_n, f_n defined by

$$e(z)=\sum_{n\in\mathbb{Z}} e_n z^{-n}=\chi(z):\exp\left(2\varphi(z)\right):,\quad f(z)=\sum_{n\in\mathbb{Z}} f_n z^{-n}=\chi(z):\exp\left(-2\varphi(z)\right):,$$

where $\chi(z)=\sum_r \chi_r z^{-r-1/2}$ and $\varphi(z)$ defined similar to (2.2) by formula:

$$\varphi(z)=\sum_{n\in\mathbb{Z}\backslash 0}\frac{h_n}{-4n}z^{-n}+\frac{h_0\log z}{4}+\widehat{Q}, \tag{3.7}$$

For the NSR algebra we consider the Fock representation of the algebra with free boson generators c_n, $n\in\mathbb{Z}\setminus\{0\}$ and free fermion generators ψ_r, $r\in\mathbb{Z}+\frac{1}{2}$

$$[c_n,c_m]=n\delta_{n+m,0},\qquad\qquad \{\psi_r,\psi_s\}=\delta_{r+s,0} \tag{3.8}$$

The NSR algebra embedded into the (completed) universal enveloping algebra of $\langle c_m,\psi_r\rangle$ by formulas:

$$\begin{aligned}
L_n&=\frac{1}{2}\sum_{k\neq 0,n}c_kc_{n-k}+\frac{1}{2}\sum_r(r-\frac{n}{2})\psi_{n-r}\psi_r+\frac{i}{2}(Qn\pm 2P)c_n\\
L_0&=\sum_{k>0}c_{-k}c_k+\sum_{r>0}r\psi_{-r}\psi_r+\frac{1}{2}(\frac{Q^2}{4}-P^2)\\
G_r&=\sum_{n\neq 0}c_n\psi_{r-n}+i(Qr\pm P)\psi_r.
\end{aligned} \tag{3.9}$$

Therefore NSR acts on the Fock representation of the algebra generated by c_n,ψ_r and the corresponding representation is isomorphic to Verma module V_Δ for general values of b,P.

After all, in free realization of representation (3.4) of algebra $\mathcal{A}(2,2)$ we have generators $w_n; h_n, D, \chi_r; c_n,\psi_r$. It follows from (3.6) that this representation has a decomposition:

$$\mathcal{F}\otimes(\mathcal{L}_{0,2}\oplus\mathcal{L}_{2,2})\otimes\mathsf{V}_\Delta=\mathcal{F}\otimes\bigoplus_{d\in\mathbb{Z}}\mathrm{F}_{2d}\otimes\mathsf{V}_\Delta. \tag{3.10}$$

Each summand on the right hand side is a representation of the subalgebra $\mathcal{H}\oplus\mathcal{H}\oplus\mathsf{F}\oplus\mathsf{NSR}$. The highest weight vectors for this subalgebra can be obtained by action of D to the vacuum. And we want to express the basis $J_{\vec{\lambda}^\sigma}$ in these generators.

3.2 The limit of the algebra

Our strategy will be the same as in the previous section: we will study the limit of $\mathcal{E}_1(q,t)$. In this case we will need level two representations of this algebra. We argue that in the limit we will get $\mathcal{A}(2,2)$ algebra.

3.2.1. The algebra $\mathcal{E}_1$ has a formal Hopf algebra structure. The formulas for the coproduct read $\Delta(C)=C\otimes C$ and

$$\begin{aligned}
\Delta(\psi^{\pm}(z))&=\psi^{\pm}\left(C_{(2)}^{\pm 1}\cdot z)\right)\cdot\psi^{\pm}\left(C_{(1)}^{\mp 1}\cdot z\right),\\
\Delta(E(z))&=E(z)\otimes 1+\psi^{-}\left(C_{(1)}\cdot z\right)\cdot E\left(C_{(1)}^2\cdot z\right),\\
\Delta(F(z))&=F(C_{(2)}^2 z)\otimes\psi^{+}(C_{(2)}z)+1\otimes F(z),
\end{aligned} \tag{3.11}$$

where $C_{(1)} := C \otimes 1$ and $C_{(2)} := 1 \otimes C$. Since we do not use the antipode and the counit in this paper, we omit them.

Using the coproduct map Δ one can define the representation of $\mathcal{E}_1$ in the space

$$\mathcal{F}_{\vec{u}} = \mathcal{F}_{u_1} \otimes \mathcal{F}_{u_2}. \tag{3.12}$$

Evidently this is a representation of the level 2. For the bosonization of this algebra we will need two Heisenberg algebras: say $a_n^{(1)}$ for $\mathcal{F}_{u_1}$ and $a_n^{(2)}$ for $\mathcal{F}_{u_2}$. For example using the coproduct formulas (3.11) and realization formulas (2.6) one can realize $E(z)$ is terms of $a_n^{(1)}$ and $a_n^{(2)}$:

$$\begin{aligned} E(z) =& u_1 : \exp\Big(-\sum_{n\in\mathbb{Z}} \frac{1-t^n}{n} a_n^{(1)} z^{-n}\Big) : + \\ &+ u_2 \exp\Big(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} (1-t^n q^{-n}) a_{-n}^{(1)} z^n \Big) :\exp\Big(-\sum_{n\in\mathbb{Z}} \frac{1-t^n}{n} (t/q)^{-n/2} a_n^{(2)} z^{-n}\Big): \end{aligned}$$

3.2.2. Now we consider the limit:

$$q = -e^{-\tau\epsilon_2},\ t = -e^{\tau\epsilon_1},\ u_i = (-1)^\sigma e^{\tau\kappa_i} \quad \tau \to 0.$$

Denote by $\mathcal{F}^{(\sigma,\sigma)}(\kappa_1,\kappa_2)$ the limit of the representation $\mathcal{F}_{u_1} \otimes \mathcal{F}_{u_2}$. We will consider the direct sum

$$\mathcal{F}^{(0,0)}(\kappa_1,\kappa_2) \oplus \mathcal{F}^{(1,1)}(\kappa_1,\kappa_2).$$

This space will have a basis labeled by pairs of two colors colored Young diagrams λ_1, λ_2 with angles colored in the same color σ. It was shown in subsubsection 3.1.2 that the character of this representation coincides with the character of (3.10).

Similar to the $r = 1$ case (formulas (2.8)) one can find in the space $\mathcal{F}^{(\sigma,\sigma)}(\kappa_1,\kappa_2)$

$$\begin{aligned} &E(z), F(z) \mapsto: (-1)^\sigma \left(\exp(2\phi^{(1)}(z)) + \exp(2\phi^{(2)}(z))\right) : +O(\tau), \\ &\frac{\psi^\pm(z) - \psi^\pm(-z)}{2} \mapsto \mp \left(2Q \sum_{\pm n\in\mathbb{Z}_{>0}} a_{2n+1} z^{-(2n+1)} \right) \tau + O(\tau^2), \\ &\frac{\psi^\pm(z) + \psi^\pm(-z)}{2} \mapsto 1 + \left(2Q^2 (\sum_{\pm n\in\mathbb{Z}_{>0}} a_{2n+1} z^{-(2n+1)})^2 - 2Q\epsilon_1 \sum_{\pm n\in\mathbb{Z}_{>0}} 2n a_{2n} z^{-2n} \right) \tau^2 + O(\tau^3), \end{aligned} \tag{3.13}$$

where

$$a_{2n+1} = a_{2n+1}^{(1)} + a_{2n+1}^{(2)}, \quad \phi^{(k)}(z) = \sum_n \frac{a_{2n+1}^{(k)}}{-2n-1} z^{-2n-1}, \ \ k = 1, 2. \tag{3.14}$$

The operators a_{2n+1} can be considered as the action of generators of $\widehat{\mathfrak{sl}}(2)_2$ (see (2.9)) on the tensor product $\mathcal{L}_{\sigma,1} \otimes \mathcal{L}_{\sigma,1}$. Then the limit of $E(z)$ coincides with the $b(z) = 2\sum_n b_n z^{-n}$, where operators b_n are the action of generators of $\widehat{\mathfrak{sl}}(2)_2$ (see (2.9)) on the same tensor product. The operators a_{2n} can be identified with Heisenberg generators w_n. In other words we have found the algebra $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_2$ in the limit.

The limit of the algebra $\mathcal{E}_1(q,t)$ should be greater then $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_2$. As was already mentioned in the introduction we conjecture that in the limit we will see the algebra $\mathcal{A}(2,2) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_2 \oplus \mathsf{NSR}$.

In order to support this conjecture one can find operators acting on the representation $\mathcal{F}^{(\sigma)}(\kappa_1) \otimes \mathcal{F}^{(\sigma)}(\kappa_1)$ which commutes with $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_2$. From the conjecture we expect to get one Heisenberg and one fermion algebra.

Indeed the Heisenberg algebra can be given as the simple difference $c_n = \frac{1}{2}(a_{2n}^{(1)} - a_{2n}^{(2)})$. Evidently these operators commute with a_n and b_n which generates $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_2$.

From the other side note that operators $a_{2n+1}^{(1)}, a_{2n+1}^{(2)}$ generate the action of $\widehat{\mathfrak{sl}}(2)_1 \oplus \widehat{\mathfrak{sl}}(2)_1$. Then the coset algebra $\dfrac{\widehat{\mathfrak{sl}}(2)_1 \oplus \widehat{\mathfrak{sl}}(2)_1}{\widehat{\mathfrak{sl}}(2)_2}$ acts on $\mathcal{L}_{\sigma,1} \otimes \mathcal{L}_{\sigma,1}$ and commutes with $\widehat{\mathfrak{sl}}(2)_2$. This coset algebra is known to be a unitary minimal model $M(3,4)$ ([26]) and then has the symmetry of the fermion algebra.

3.2.3. In order to recapitulate the results of the previous subsubsection we present a formulas for the bosonization $w_n; h_n, \chi_r; c_n, \psi_r$, in terms of two limit Heisenberg algebras $a_n^{(1)}$ and $a_n^{(2)}$.

$$w_n = a_{2n}^{(1)} + a_{2n}^{(2)}, \qquad c_n = \frac{a_{2n}^{(1)} - a_{2n}^{(2)}}{2} \tag{3.15}$$

$$\sum_n h_n z^{-2n} - 1 = \frac{(-1)^{\sigma+1}}{4} \left(\exp(2\phi^{(1)}) + \exp(2\phi^{(2)}) + \exp(-2\phi^{(1)}) + \exp(-2\phi^{(2)})\right) \tag{3.16}$$

$$\sum_r \chi_r z^{-2r} = (-1)^{\sigma+1} \frac{i}{2\sqrt{2}} \left(\exp(\phi^{(1)} + \phi^{(2)}) - \exp(-\phi^{(1)} - \phi^{(2)})\right), \tag{3.17}$$

$$\sum_r \psi_r z^{-2r} = \frac{i}{2\sqrt{2}} \left(\exp(\phi^{(1)} - \phi^{(2)}) - \exp(-\phi^{(1)} + \phi^{(2)})\right), \tag{3.18}$$

where fields $\phi^{(k)}$, $k = 1, 2$ were introduced in (3.14). The explanation of formulae for fermions χ_r, ψ_r is given in Appendix C. Anyway it is easy to check that commutation relations of r. h. s. are correct.

3.3 Basis and the vertex operator for algebra $\mathcal{E}_1(q,t)$

3.3.1. The commutative subalgebra of the algebra $\mathcal{E}_1(q,t)$ acts on the space $\mathcal{F}_{u_1} \otimes \mathcal{F}_{u_2}$. There exists a certain basis $P_{\vec{\lambda}}$ which consists of eigenvectors of the operator E_0:

$$E_0 |P_{\vec{\lambda}}\rangle = \varepsilon_{\vec{\lambda},\vec{u}} |P_{\vec{\lambda}}\rangle, \qquad \varepsilon_{\vec{\lambda},\vec{u}} := u_1 \varepsilon_{\lambda^{(1)}} + u_2 \varepsilon_{\lambda^{(2)}}. \tag{3.19}$$

Here $\vec{\lambda} = (\lambda_1, \lambda_2)$ is a pair of partitions. Similar to Macdonald polynomials this basis can be defined as orthogonalizaiton of certain monomial basis. The first examples of vectors $P_{\vec{\lambda}}$ have the form ([10]):

$$\begin{aligned}
&P_{((1),\varnothing)} = P_{(1)} \otimes 1, \qquad P_{(\varnothing,(1))} = 1 \otimes P_{(1)} + (q/t)^{1/2} \frac{(t-q)u_2}{q(u_1 - u_2)} P_{(1)} \otimes 1 \\
&P_{((1,1,1),\varnothing)} = P_{(1,1,1)} \otimes 1, \qquad P_{((2,1),\varnothing)} = P_{(2,1)} \otimes 1, \qquad P_{((3),\varnothing)} = P_{(3)} \otimes 1, \\
&P_{((1,1),(1))} = P_{(1,1)} \otimes P_{(1)} + (q/t)^{1/2} \frac{(t-q)u_2}{q(qu_1 - u_2)} P_{(2,1)} \otimes 1 + (q/t)^{1/2} \frac{(1-q)(t-q)(1-t^3)t^2 u_2}{q(1-qt^2)(1-t)(u_1 - t^2 u_2)} P_{(1,1,1)} \otimes 1, \\
&P_{((2),(1))} = P_{(2)} \otimes P_{(1)} + (q/t)^{1/2} \frac{(t-q)u_2}{q(q^2 u_1 - u_2)} P_{(3)} \otimes 1 + (q/t)^{1/2} \frac{(1-q^2)(t-q)(1-qt^2)tu_2}{q(1-qt)(1-q^2 t)(u_1 - tu_2)} P_{(2,1)} \otimes 1,
\end{aligned} \tag{3.20}$$

where P_λ denotes Macdonald polynomials (B.2).

It follows from the definition of $P_{\vec{\lambda}}$ that the basis $\{P_{\vec{\lambda}}\}$ is orthogonal. It is clear from the coproduct formula (3.11) that all vectors $P_{\lambda_1,\varnothing}$ have the form $P_{\lambda_1}\otimes 1$.

In was conjectured in [10] that there exists another normalization $K_{\vec{\lambda}}$ which is more similar to the integral form of Macdonald polynomials $J_\lambda(q,t)$. The norms in this normalization have nice factorized form:

$$\langle K_{\vec{\lambda}}|K_{\vec{\lambda}}\rangle = \big((-1)^m(t/q)^{m-1}u\big)^{|\vec{\lambda}|}\times\prod_{k=1}^m u_k^{-(m-2)|\lambda^{(k)}|}q^{-(m-2)n(\lambda^{(k)\prime})}t^{(m-2)n(\lambda^{(k)})}\times \prod_{i,j=1}^m N_{\lambda^{(i)},\lambda^{(j)}}(qu_i/tu_j), \tag{3.21}$$

where $u=u_1u_2$ and $N_{\lambda,\mu}$ is defined in (2.23).

3.3.2. Vertex operator $\Phi(z)\colon \mathcal{F}_{\vec{u}}\to\mathcal{F}_{\vec{v}}$ is defined by the commutation relations (2.20), where $u=u_1u_2$ and $v=v_1v_2$.

Conjecture 3.1 ([10])**.** *(1) The operator $\Phi(z)$ exists uniquely.*
(2)The matrix elements of operator $\Phi(z)$ in the basis $K_{\vec{\lambda}}$ have factorized (Nekrasov) form

$$\langle K_{\vec{\lambda}}|\Phi(z)|K_{\vec{\mu}}\rangle = \left(\frac{t^2}{q^2}uv\right)^{|\vec{\lambda}|}\left(\frac{t}{q}v\right)^{-|\vec{\mu}|} z^{|\vec{\lambda}|-|\vec{\mu}|}\times \prod_{k=1}^2 v_k^{-|\lambda_k|}u_k^{|\mu_k|}q^{-n(\lambda_k'+n(\mu_k'))}q^{n(\lambda_k-n(\mu_k))}\prod_{i,j=1}^2 N_{\lambda^{(i)},\mu^{(j)}}\left(\frac{qv_i}{tu_j}\right), \tag{3.22}$$

where $u=u_1u_2$, $v=v_1v_2$, $N_{\lambda,\mu}$ is defined in (2.23) *and $n(\lambda)$ is defined in* (B.1)*.*

One can see that in the limit $q,t\to -1$ of (3.22) we get the desired expression (1.2) (up to some sign and factors like $2^{|\lambda^\diamond|}$). So it is natural to consider the limit of the basis $K_{\vec{\lambda}}$.

We remark that in the level one case the basis K_λ differs from the basis J_λ by scalar factor. But this factor is a some power of parameters t,q,u so it is not very important for our limit.

3.4 Basis for the algebra $\mathcal{A}(2,2)$

3.4.1. We will study the basis in the representation

$$\mathcal{F}^{(0,0)}(\kappa_1,\kappa_2)\oplus\mathcal{F}^{(1,1)}(\kappa_1,\kappa_2)=\mathcal{F}\otimes(\mathcal{L}_{0,2}\oplus\mathcal{L}_{2,2})\otimes\mathsf{V}_\Delta \tag{3.23}$$

of the algebra $\mathcal{A}(2,2)$. This representation is called the Neveu–Schwarz representation of the algebra $\mathcal{A}(2,2)$. The equality (3.23) can be considered as the isomorphism between two realizations of the same representation. On the left hand side the algebra $\mathcal{A}(2,2)$ arises as the limit of $\mathcal{E}_1(q,t)$, see subsubsection 3.2.2. On the right hand side we use realization $\mathcal{H}\oplus\widehat{\mathfrak{sl}}(2)_2\oplus\mathsf{NSR}$.

The continuous parameters κ_1,κ_2 on the left hand side of (3.23) and Δ_{NS} on the right hand side are related in the realization of NSR in the limit of $\mathcal{E}_1(q,t)$. We suggest the relation $P=\frac{1}{2}(\kappa_2-\kappa_1)$, where $\Delta_{\mathsf{NS}}=\frac{1}{2}(Q^2/4-P^2)$. We denote by $|P\rangle$ the vacuum vector of $\mathcal{F}\otimes(\mathcal{L}_{0,2}\oplus\mathcal{L}_{2,2})\otimes\mathsf{V}_\Delta$. This vector coincides with $|\kappa_1,\kappa_2\rangle^{0,0}$ the vacuum vector

of $\mathcal{F}^{(0,0)}(\kappa_1,\kappa_2)$. The vacuum vector of $\mathcal{F}^{(1,1)}(\kappa_1,\kappa_2)$ can be obtained by action of shift operator D: $|\kappa_1,\kappa_2\rangle^{1,1} = D|P\rangle$.

By the limit procedure the space (3.23) has a basis $J^{(2)}_{\vec{\lambda}^\sigma}$, labeled by $\sigma = 0,1$ and $\vec{\lambda} = (\lambda_1,\lambda_2)$. The desired basis $J^{(2)}_{\vec{\lambda}^\sigma}$ respects the grading by h_0 and L_0 as in (3.10). For example the action of h_0 can be deduced as a limit of the action E_0 (3.19) as in $r=1$ case see (2.17). From the decomposition (3.10) follows that the space with the given h_0 forms a representation of algebra $\mathcal{H}\oplus\mathcal{H}\oplus\mathsf{F}\oplus\mathsf{NSR}$. All the highest vectors for the decomposition (3.10) have the form $D^k|P\rangle$, e.g:

$$J^{(2)}_{((1)^1,(1)^1)} = D^{-1}|P\rangle, \quad J^{(2)}_{(\varnothing^0,\varnothing^0)} = |P\rangle, \quad J^{(2)}_{(\varnothing^1,\varnothing^1)} = D|P\rangle, \quad J^{(2)}_{((1)^0,(1)^0)} = D^2|P\rangle. \tag{3.24}$$

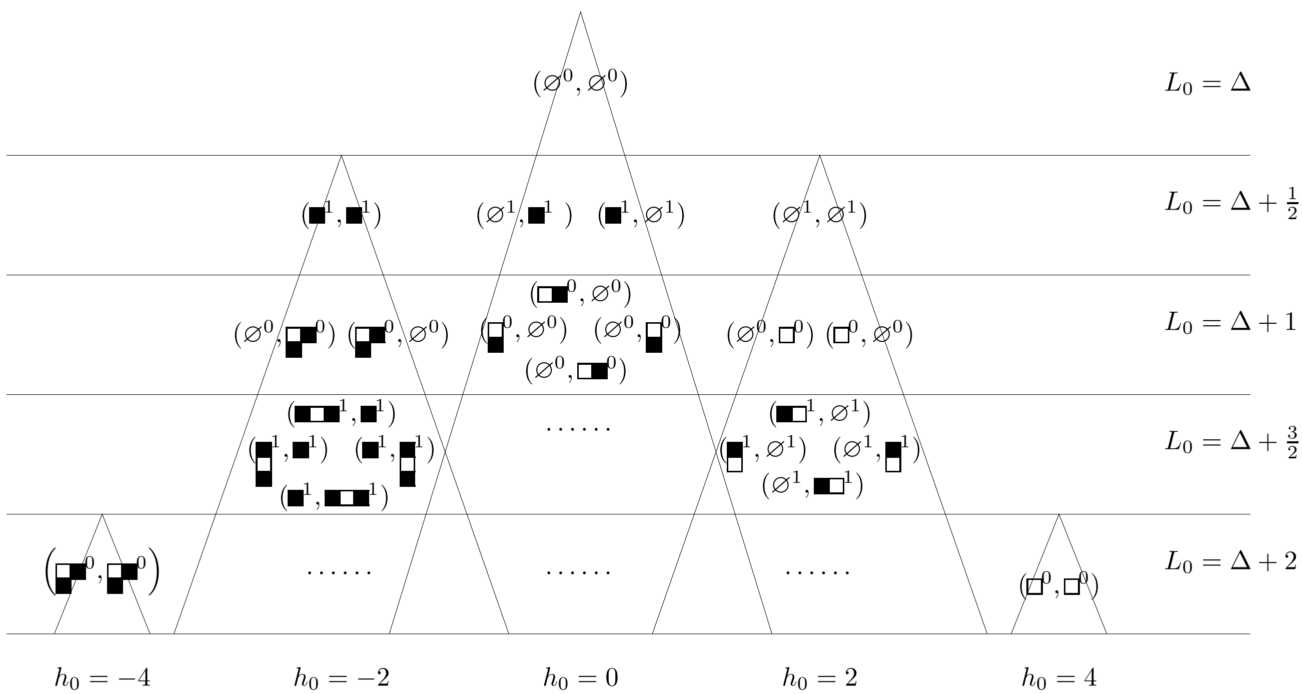

Figure 3: The basis in $\mathcal{F}^{(0,0)}(\kappa_1,\kappa_2)\oplus\mathcal{F}^{(1,1)}(\kappa_1,\kappa_2)$. Pair of colored diagrams $(\lambda_1^\sigma,\lambda_2^\sigma)$ represents a vector $J^{(2)}_{\vec{\lambda}^\sigma}$. The interior of each angle corresponds to the representation $\mathcal{F}\otimes\mathrm{F}_{2d}\otimes\mathsf{V}_\Delta\subset\mathcal{F}^{(0,0)}(\kappa_1,\kappa_2)\oplus\mathcal{F}^{(1,1)}(\kappa_1,\kappa_2)$.

Taking the limit of formulas for the $P_{\vec{\lambda}}$ in terms of $a_n^{(1)}$ and $a_n^{(2)}$ (like (3.20)) and (3.15)-(3.18) one can find the first examples:

$$\begin{gathered}
J^{(2)}_{(\varnothing^0,\varnothing^0)} = |P\rangle, \quad J^{(2)}_{((1)^1,\varnothing^1)} = \left(G_{-1/2}+i\frac{Q+2P}{2}\chi_{-1/2}\right)|P\rangle, \quad J^{(2)}_{((1)^1,\varnothing^1)} = \left(G_{-1/2}+i\frac{Q-2P}{2}\chi_{-1/2}\right)|P\rangle,\\
J^{(2)}_{((2)^0,\varnothing^0)} = \left(-2L_{-1}-\frac{2i}{b}\chi_{-1/2}G_{-1/2}-\frac{i(Q+2P)}{2}w_{-1}-\frac{Q+2P}{2b}h_{-1}\right)|P\rangle,\\
J^{(2)}_{((1,1)^0,\varnothing^0)} = \left(-2L_{-1}-2bi\chi_{-1/2}G_{-1/2}-\frac{i(Q+2P)}{2}w_{-1}-\frac{b(Q+2P)}{2}h_{-1}\right)|P\rangle,\\
J^{(2)}_{(\varnothing^0,(2)^0)} = \left(-2L_{-1}-\frac{2i}{b}\chi_{-1/2}G_{-1/2}-\frac{i(Q-2P)}{2}w_{-1}-\frac{Q-2P}{2b}h_{-1}\right)|P\rangle,
\end{gathered}\tag{3.25}$$

$$J^{(2)}_{((2,2)^0,\varnothing^0)} = \Big(4L_{-1}^2 - 4G_{-3/2}G_{-1/2} - 2Q(Q+2P)L_{-2} + i4(Q+P)w_{-1}L_{-1} - \frac{(Q+P)(Q+2P)}{2}w_{-1}^2 + \\ + \frac{(Q+P)(Q+2P)}{2}h_{-1}^2 - 2(Q+P)(Q+2P)\chi_{-3/2}\chi_{-1/2} - iQ(Q+P)(Q+2P)w_{-2} + \\ + i4(Q+P)h_{-1}\chi_{-1/2}G_{-1/2}\Big)|P\rangle;$$

The basis $J^{(2)}_{\vec{\lambda}^\sigma}$ is orthogonal and normalization chosen such that:

$$\langle J^{(2)}_{\vec{\lambda}^\sigma}|J^{(2)}_{\vec{\lambda}^\sigma}\rangle = \begin{cases} \prod_{i,j=1}^2 N^{(2)}_{\lambda_i^\sigma,\lambda_j^{\tilde{\sigma}}}(P_i - P_j) & \text{if } |\vec{\lambda}| \equiv 0 \mod 2 \\ \frac{1}{2}\prod_{i,j=1}^2 N^{(2)}_{\lambda_i^\sigma,\lambda_j^{\tilde{\sigma}}}(P_i - P_j) & \text{if } |\vec{\lambda}| \equiv 1 \mod 2, \end{cases}$$

where $(P_1, P_2) = (P, -P)$ and $N^{(2)}_{\lambda^\sigma,\mu^{\tilde{\sigma}}}(\alpha)$ defined in (2.26).

In the remaining part of this section we will state some properties of this basis. For simplicity we will consider only the subspace $\mathcal{F} \otimes \mathrm{F}_0 \otimes \mathsf{V}_\Delta$ i.e subspace where $h_0 = 0$. In combinatorial terms it means that

$$\Big(\sigma = 0,\;\; N_0(\vec{\lambda}) = N_1(\vec{\lambda})\Big) \quad \text{or} \quad \Big(\sigma = 1,\;\; N_1(\vec{\lambda}) = N_0(\vec{\lambda}) + 1\Big).$$

3.4.2. Elements of the form $J^{(2)}_{\lambda^\sigma,\varnothing^\sigma}$ can be given in explicit form. After bosonization of the basis elements (using sign " + " in formula (3.9)) we have expressions like:

$$J^{(2)}_{((1)^0,\varnothing^0)} = \frac{i(Q+2P)}{2}\left(\chi_{-1/2} + \psi_{-1/2}\right)|P\rangle,$$
$$J^{(2)}_{((2)^0,\varnothing^0)} = -(Q+2P)\left(i(\frac{1}{2}w_{-1} + c_{-1}) + b^{-1}(\frac{1}{2}h_{-1} + \chi_{-1/2}\psi_{-1/2})\right)|P\rangle,$$
$$J^{(2)}_{((1,1)^0,\varnothing^0)} = -(Q+2P)\left(i(\frac{1}{2}w_{-1} + c_{-1}) + b(\frac{1}{2}h_{-1} + \chi_{-1/2}\psi_{-1/2})\right)|P\rangle.$$

Recall that in deformed case basis vectors $P_{\lambda,\varnothing} = P_\lambda \otimes 1$ (see 3.3.1)). So in the limit the basis vectors $J^{(2)}_{\lambda^\sigma,\varnothing^\sigma}$ are identified to the Uglov polynomials J_λ of the first Heisenberg algebra $a_n^{(1)}$ up to some normalization factor depending on P. Let $\Omega_\lambda(P)$ denote

$$\Omega_\lambda(P) = \prod_{s\in\lambda,\; i+j\equiv 0 \bmod 2} (2P + ib^{-1} + jb). \tag{3.26}$$

Proposition 3.1. *(1) Let $\sigma = 0$, $N_0(\vec{\lambda}) = N_1(\vec{\lambda})$. After bosonization we have:*

$$J^{(2)}_{\lambda^\sigma,\varnothing^\sigma} = \Omega_\lambda(P)J^{(2)}_\lambda(a_k^{(1)}),$$

where $J^{(2)}_\lambda$ defined in (2.14),(2.15). *The polynomial $J^{(2)}_\lambda$ can also be rewritten in terms of two Heisenberg algebras $h_n^{(1)}$ and $w_n^{(1)}$ which can be found from $a_n^{(1)}$ as in* (2.11)*:*

$$w_n^{(1)} = a_{2n}^{(1)}, \qquad \sum h_n^{(1)} z^{-2n} = \frac{1}{2} + \frac{(-1)^{\sigma+1}}{4}\left(\exp(2\phi^{(1)}) + \exp(-2\phi^{(1)})\right).$$

(2) Let $\sigma = 1$, $N_1(\vec{\lambda}) = N_0(\vec{\lambda}) + 1$. After bosonization we have:

$$J^{(2)}_{\lambda^\sigma,\varnothing^\sigma} = \frac{-1}{\sqrt{2}}\Omega_\lambda(P)J^{(2)}_\lambda(a_k^{(1)}),$$

where $J^{(2)}_\lambda$ defined in (2.14),(2.15).

Operators $h_n^{(1)}$ and $w_n^{(1)}$ used in Proposition 3.1 can be expressed by use of (3.15)-(3.18):

$$w_n^{(1)} = \frac{1}{2}w_n + c_n, \qquad h_n^1 = \frac{1}{2}h_n - \sum \chi_r\psi_{n-r}.$$

One can use another bosonization another free field representation which corresponds to the sign "$-$" in formula (3.9)). Denote the corresponding generators by $\tilde{c}_n, \tilde{\psi}_r$. In this case the vectors $J^{(2)}_{\varnothing^\sigma,\lambda^\sigma}$ are identified with Uglov polynomials J_{λ^σ} depending on $\tilde{a}_n^{(1)}$, with factor $\Omega_\lambda(-P)$.

In the case $Q = b + b^{-1} = 0$ the situation simplifies more. It follows from (3.9)) that in this case $\tilde{c}_k = -c_k$ and $\tilde{\psi}_r = -\psi_r$. Therefore $\tilde{a}_n^{(1)} = a_n^{(2)}$ and generic vectors $J_{\lambda_1^\sigma,\lambda_2^\sigma}$ become a product of two Uglov polynomials $J_{\lambda_1^\sigma}(a_n^{(1)}) \cdot J_{\lambda_2^\sigma}(a_n^{(2)})$. The analogous fact for the algebra $\mathcal{A}(1,2) = \mathcal{H} \oplus \mathrm{Vir}$ was given in [28].

The corresponding Heisenberg generators $h_n^{(2)}$ and $w_n^{(2)}$ read:

$$w_n^{(2)} = a_{2n}^{(2)}, \qquad \sum h_n^{(2)} z^{-2n} - \frac{1}{2} = \frac{(-1)^{\sigma+1}}{4}\left(\exp(2\phi^{(2)}) + \exp(-2\phi^{(2)})\right);$$
$$w_n^{(2)} = \frac{1}{2}w_n - c_n, \qquad h_n^{(2)} = \frac{1}{2}h_n + \sum \chi_r\psi_{n-r}.$$

3.4.3. It remains to take the limit of the vertex operator $\Phi(z)$ for the algebra $\mathcal{E}_1(q,t)$. Since we consider the representation (3.23) then we will take the limit of two vertex operators:

$$\Phi_0(z)\colon \mathcal{F}^{(0,0)}(\kappa_1,\kappa_2) \to \mathcal{F}^{(0,0)}(\tilde{\kappa}_1,\tilde{\kappa}_2), \qquad \Phi_0(z)\colon \mathcal{F}^{(1,1)}(\kappa_1,\kappa_2) \to \mathcal{F}^{(1,1)}(\tilde{\kappa}_1,\tilde{\kappa}_2),$$
$$\Phi_1(z)\colon \mathcal{F}^{(0,0)}(\kappa_1,\kappa_2) \to \mathcal{F}^{(1,1)}(\tilde{\kappa}_1,\tilde{\kappa}_2), \qquad \Phi_1(z)\colon \mathcal{F}^{(1,1)}(\kappa_1,\kappa_2) \to \mathcal{F}^{(0,0)}(\tilde{\kappa}_1,\tilde{\kappa}_2).$$

Similar to $r = 1$ case norms (3.21) and matrix elements (3.22) tend to 0 when $\tau \to 0$. Combinatorial considerations suggest that in the case $\sigma = \tilde{\sigma}$ we have:

$$\lim_{\tau\to 0}\left(\frac{\langle K_{\vec{\lambda}}|\Phi(z)|K_{\vec{\mu}}\rangle}{\sqrt{\langle K_{\vec{\lambda}}K_{\vec{\lambda}}\rangle}\cdot\sqrt{\langle K_{\vec{\mu}}K_{\vec{\mu}}\rangle}}\right) = \begin{cases}\neq 0 & \text{if } h_0 \text{ have the same eigenvalues on } J^{(2)}_{\vec{\lambda}^\sigma} \text{ and } J^{(2)}_{\vec{\mu}^{\tilde{\sigma}}}\\ 0, & \text{otherwise}\end{cases}$$

In the case $|\sigma - \tilde{\sigma}| = 1$ we have:

$$\lim_{\tau\to 0}\left(\frac{\tau\langle K_{\vec{\lambda}}|\Phi(z)|K_{\vec{\mu}}\rangle}{\sqrt{\langle K_{\vec{\lambda}}K_{\vec{\lambda}}\rangle}\cdot\sqrt{\langle K_{\vec{\mu}}K_{\vec{\mu}}\rangle}}\right) = \begin{cases}\neq 0 & \text{if } h_0 \text{ have the same eigenvalues on } J^{(2)}_{\vec{\lambda}^\sigma} \text{ and } J^{(2)}_{\vec{\mu}^{\tilde{\sigma}}}\\ 0, & \text{otherwise}\end{cases}$$

Therefore in the leading order of the limit of operators $\Phi_0(z)$ and $\Phi_1(z)$ we obtain the operator which commute with h_0. We suggest that this limit coincides with the vertex operator

$$\mathcal{V}_\alpha(z)\cdot\Phi_\alpha^{\mathsf{NS}}(z).$$

Here $\mathcal{V}_\alpha$ is a standard rotated vertex operator on Heisenberg algebra $\mathcal{H}$:

$$\mathcal{V}_\alpha(z) = \exp\left(i(\alpha - Q)\sum_{n=1}^{\infty}\frac{w_{-n}z^n}{2n}\right)\exp\left(i\alpha\sum_{n=1}^{\infty}\frac{w_n z^{-n}}{-2n}\right)$$

and $\Phi_\alpha^{\mathsf{NS}}$ is the primary field of the NSR algebra with conformal dimension $\Delta(\alpha) = \frac{1}{2}\alpha(Q-\alpha)$, $\Psi_\alpha^{\mathsf{NS}}$ its super partner with the dimension $\Delta(\alpha)+1/2$. These operators can be defined

by the commutation relations:

$$
\begin{aligned}
&[L_n, \Phi_\alpha^{\mathsf{NS}}] = (z^{n+1}\partial_z + (n+1)\Delta(\alpha)z^n)\Phi_\alpha^{\mathsf{NS}},\\
&[L_n, \Psi_\alpha^{\mathsf{NS}}] = (z^{n+1}\partial_z + (n+1)(\Delta(\alpha)+1/2)z^n)\Psi_\alpha^{\mathsf{NS}},\\
&[G_r, \Phi_\alpha^{\mathsf{NS}}] = z^{r+1/2}\Psi_\alpha^{\mathsf{NS}},\\
&\{G_r, \Psi_\alpha^{\mathsf{NS}}\} = (z^{r+1/2}\partial_z + (2r+1)\Delta(\alpha)z^{r-1/2})\Phi_\alpha^{\mathsf{NS}},
\end{aligned}
\tag{3.27}
$$

Proposition 3.2. *Consider two colored partitions $\lambda^{\tilde{\sigma}}$, μ^{σ} such that h_0 gradings on $J^{(2)}_{\vec{\lambda}^{\tilde{\sigma}}}$ and $J^{(2)}_{\vec{\mu}^{\sigma}}$ coincide. Then matrix elements of the vertex operator $\mathcal{V}_\alpha(z)\cdot\Phi_\alpha^{NS}(z)$ have completely factorized form:*

$$
\langle J^{(2)}_{\vec{\lambda}^{\tilde{\sigma}}}|\mathcal{V}_\alpha(z)\cdot\Phi_\alpha^{NS}(z)|J^{(2)}_{\vec{\mu}^{\sigma}}\rangle = \begin{cases}
\prod_{i,j=1}^2 N^{(2)}_{\lambda_i^{\tilde{\sigma}},\mu_j^{\sigma}}(\alpha+P_i-P'_j) & \textit{if } \tilde{\sigma}=0\\
-\prod_{i,j=1}^2 N^{(2)}_{\lambda_i^{\tilde{\sigma}},\mu_j^{\sigma}}(\alpha+P_i-P'_j) & \textit{if } \tilde{\sigma}=1, \sigma=0\\
\frac{1}{2}\prod_{i,j=1}^2 N^{(2)}_{\lambda_i^{\tilde{\sigma}},\mu_j^{\sigma}}(\alpha+P_i-P'_j) & \textit{if } \tilde{\sigma}=1, \sigma=1,
\end{cases}
\tag{3.28}
$$

where $N^{(2)}_{\lambda^{\tilde{\sigma}},\mu^{\sigma}}(\alpha)$ defined in (2.26).

This proposition was checked up to the level 2. This proposition provides very nontrivial evidence that at the limit $q,t\to -1$ of $\mathcal{E}_1(q,t)$ we will have primary fields of the Neveu–Schwarz–Ramond algebra.

4 Singular vectors

The Uglov symmetric polynomials are natural generalization of the Jack symmetric polynomials. The singular vectors of the Virasoro algebra have the remarkable description in terms of Jack polynomials [16, 17]. Uglov polynomials have the similar application to the representation theory of the Neveu–Schwarz–Ramond algebra.

We assume that parameter b is generic. Let $m,n\in\mathbb{Z}$ such that $m-n\equiv 0 \mod 2$, and

$$
P_{m,n} = -(mb^{-1}+nb)/2. \tag{4.1}
$$

Then for $P=P_{m,n}$ there exists operator $D_{m,n}$ such that vector $D_{m,n}|P_{m,n}\rangle$ is singular i.e.

$$
L_k D_{m,n}|P_{m,n}\rangle = G_r D_{m,n}|P_{m,n}\rangle = 0,\ k,r>0 \qquad L_0 D_{m,n}|P_{m,n}\rangle = (\Delta_{m,n}+\frac{mn}{2})D_{m,n}|P_{m,n}\rangle,
$$

where $\Delta(P)=\frac{1}{2}(Q^2/4-P^2)$. The vector $D_{m,n}|P_{m,n}\rangle$ generates a submodule in $\mathsf{V}_{\Delta_{m,n}}$. For $P=P_{m,n}$ the operator $D_{m,n}$ is unique up to normalization, for other values of P the Verma module V_Δ is irreducible [29, 30].

First examples of singular vectors $D_{m,n}|P_{m,n}\rangle$ have the form:

$$
\begin{aligned}
&D_{1,1}|P_{1,1}\rangle = G_{-1/2}|P_{1,1}\rangle,\\
&D_{3,1}|P_{1,3}\rangle = (L_{-1}G_{-1/2}+b^2G_{-3/2})|P_{1,3}\rangle,\\
&D_{2,2}|P_{2,2}\rangle = (L_{-1}^2+\frac{1}{2}Q^2L_{-2}-G_{-3/2}G_{-1/2})|P_{2,2}\rangle.
\end{aligned}
$$

Expressing these $D_{m,n}$ through free field generators $\tilde{c}_k,\tilde{\psi}_k$ (using sign " $-$ " in formula

(3.9)) we get:

$$\begin{aligned}
&D_{1,1}|P_{1,1} = -iQ\tilde{\psi}_{-1/2}|P_{1,1}\rangle,\\
&D_{3,1}|P_{1,3}\rangle = -Q(Q+2b)(\tilde{c}_{-1}\tilde{\psi}_{-1/2}+ib\tilde{\psi}_{-3/2})|P_{1,3}\rangle,\\
&D_{2,2}|P_{2,2}\rangle = -2Q^2(\tilde{c}_{-1}^2+\frac{i}{2}Q\tilde{c}_{-2}-2\tilde{\psi}_{-3/2}\tilde{\psi}_{-1/2})|P_{2,2}\rangle.
\end{aligned}$$

Proposition 4.1. *Denote by* (m^n) *the rectangular Young diagram* $(\underbrace{m,m,\dots,m}_{n})$. *The operators* $D_{m,n}$ *coincide (up to normalization) with the Uglov polynomials* $J^{(2)}_{(m^n)}$ *defined in* (2.14),(2.15) *with identification:*

$$\tilde{c_n} = \frac{1}{2}a_{2n}, \qquad \sum_r \tilde{\psi}_r z^{-2r} = \frac{i}{2\sqrt{2}}\left(\exp(\phi^-+2\phi^+)-\exp(-\phi^--2\phi^+)\right), \tag{4.2}$$

where $\phi^+(z)=\sum_{n\in\mathbb{Z}_{>0}}\dfrac{a_{2n+1}}{-2n-1}z^{-2n-1}$, $\phi^-(z)=\sum_{n\in\mathbb{Z}_{<0}}\dfrac{a_{2n+1}}{-2n-1}z^{-2n-1}$.

We have checked this Proposition up to the level 9/2. This Proposition is one of the main results of the paper.

The singular vectors related to the basis $J^{(2)}_{\lambda_1^\sigma,\lambda_2^\sigma}$ as follows. If we put $P=P_{m,n}$ then the vectors $J^{(2)}_{\lambda_1^\sigma,\varnothing^\sigma}$ vanishes after c_n,ψ_r free field realization due to vanishing of the factor $\Omega_{m\times n}(P_{m,n})$ (3.26) (see Proposition 3.1). Therefore, the vector $J^{(2)}_{\lambda_1^\sigma,\varnothing^\sigma}$ belongs to the kernel of the natural map from the Verma module to the Fock module. Then the vector $J^{(2)}_{\lambda_1^\sigma,\varnothing^\sigma}$ is singular for the algebra $\mathcal{A}(2,2)=\mathcal{H}\oplus\widehat{\mathfrak{sl}}(2)_2\oplus\mathsf{NSR}$.

The algebras $\mathcal{H}$ and $\widehat{\mathfrak{sl}}(2)_2$ have no singular vectors so $J^{(2)}_{\lambda_1^\sigma,\varnothing^\sigma}$ is singular vector for the NSR algebra. Therefore if we put $P=P_{m,n}$ then all terms depending on w_n,h_n,χ_r vanish and the vector will depend only on L_n,ψ_r. One can easily see this on the examples (3.25). If we use another bosonization we will get the vector depending only on $\tilde{c}_n,\tilde{\psi}_r$. The proposition 4.1 means that this vector can be described in terms of Uglov polynomials.

It is worth to note that in the recent paper [31] Desrosiers, Lapointe and Mathieu propose an another expression for the singular vectors $D_{m,n}$. In the formula in loc. cit. $D_{m,n}$ become a linear combination of the *superJack polynomials* contrary to our result where the only one *Uglov polynomial* $J^{(2)}_{(m^n)}$ is used. From the other hand expressions from loc. cit. do not involve nonpolynomial change of variables as in Proposition 4.1.

5 Concluding remarks

5.1. In the paper we dealt only $p=2$ case but the methods looks quite general. It is natural to expect that in the limit the limit $q,t\to\sqrt[p]{1}$ of the representations of quantum toroidal $\mathfrak{gl}(1)$ algebra $\mathcal{E}_1(q,t)$ on get the representation of the algebra $\mathcal{A}(r,p)$. For example limit of matrix elements (3.21) for $\mathcal{E}_1(q,t)$ coincide with matrix elements (1.2) for $\mathcal{A}(r,p)$. The rank p Uglov polynomials form a natural basis in the representation of the algebra $\mathcal{A}(1,p)=\mathcal{H}\oplus\widehat{\mathfrak{sl}}(p)_1$ [13].

It would be interesting to give a mathematically rigorous proof of our construction of the algebra $\mathcal{A}(r,p)$. Note that this limit construction should give principal (Lepowsky-Wilson) construction of this algebra. It is expected that this algebra is a subalgebra of more general Yangian algebra.

It worth to note some relevant works. Ten years ago Hara et al [32] considered the limit of deformed Virasoro algebra where $q \to 1$, $t \to -1$ and observe in the limit the Lepowsky–Wilson realization of Verma module of the algebra $\widehat{\mathfrak{sl}}(2)$. This limit can be considered as limit of $r = 2$ representations of $\mathcal{E}_1(q,t)$. From the geometric side of the AGT relation the limit algebra corresponds to the equivariant integration on the affine Laumon spaces. In terms of gauge theory this corresponds to the presence of the surface operators [3].

Note also that Taro Kimura used the limit when q, t go to the root of 1 in the matrix model which corresponds to the instanton counting on the ALE space [33] and more general toric singularities [34].

5.2. In the $r = 2$ case we have constructed only vertex which doesn't depend on $\widehat{\mathfrak{sl}}(2)_2$ part. In particular the equations (3.28) do not determine the basis $J^{(2)}_{\vec{\lambda}^{\sigma}}$, therefore we used the deformed algebra. It would be interesting to find the formulas similar (3.28) for more general vertex operators.

One of the natural approaches is to add the Ramond sector to the considerations. It expected that our construction works in the Ramond representation of algebra $\mathcal{A}(2,2) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(2)_2 \oplus \mathsf{NSR}$ and the resulting basis factorizes the vertex operator which maps Neveu–Schwarz representation to Ramond one and back. This vertex should have the form like $\mathcal{V}_\alpha \cdot \Phi_{1/2} \cdot R$. Here R is Ramond primary field and $\Phi_{1/2}$ correspond to highest weight vector in $\mathcal{L}_{0,2}$ by the operator-state correspondence. This operator doesn't commute with h_0 (on the instanton side changes the first Chern class c_1). Some checks for this vertex were made in [35].

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Appendix A. Partitions

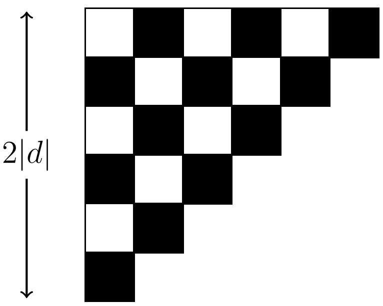
We basically follow [36] for the notations. A partition λ is a series of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots$ with finitely many nonzero entries, $|\lambda| := \sum_{i \geq 1} \lambda_i$. If $\lambda_l > 0$ and $\lambda_{l+1} = 0$, we write $\ell(\lambda) := l$ and call it the length of λ. We identify the partition and the corresponding Young diagram.

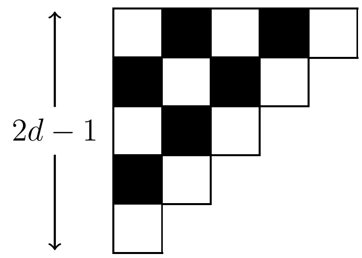
The conjugate partition of λ is denoted by λ' which corresponds to the transpose of the diagram λ. The empty partition is denoted by $\varnothing$. The dominance ordering is defined by $\lambda \geq \mu \Leftrightarrow |\lambda| = |\mu|$ and $\sum_{k=1}^{i} \lambda_k \geq \sum_{k=1}^{i} \mu_k$ for all $i = 1, 2, \ldots$.

In Section 2 and later we use the colouring of Young diagrams in two colors, labeled by residues modulo 2: the box $s \in \lambda$ with coordinates (i,j) has color $i - j + \sigma$. Here

$\sigma = 0, 1$ is a color of the angle. We will denote such colored Young diagram by λ^σ in order to stress the coloring. By $N_i(\lambda^\sigma)$ we denote the number of boxes of color i, $i = 0, 1$. Let $d(\lambda^\sigma) = N_0(\lambda^\sigma) - N_1(\lambda^\sigma)$. The following lemmas are standard.

Lemma A.1. *Let $\sigma = 0$. There is a smallest partition μ with $d(\mu) = d$. This partition consists of $2d^2 - d$ boxes and has a "triangular" form with edge length $2|d|$ for $d \leq 0$ and $2d - 1$ for $d > 0$.*





If $\sigma = 1$ then the smallest partition μ with $d(\mu) = d$ has $2d^2 + d$ boxes.

Such minimal diagram is called 2-core ([36, Sec 1.1 Ex. 8]). For any partition λ its 2-core is denoted by $\widetilde{\lambda}$ and can be obtained as follows. Remove the rectangle 1×2 (or 2×1) from the diagram of λ in such a way that what remains is the diagram of a partition, and continue removing such rectangles in this way as long as possible. The result of this process is the 2-core $\widetilde{\lambda}$ of λ and the result is independent of the sequence of removals.

Lemma A.2. *The number of partitions λ such that $d(\lambda) = d$ and $|\lambda| - |\widetilde{\lambda}| = 2n$ equals to the number of pairs of partitions (μ_1, μ_2) such that $|\mu_1| + |\mu_2| = n$.*

This Lemma has a bijective proof (see e.g. [36, Sec 1.1 Ex. 8]). In terms of generating functions we have that

$$\chi^{(1)}_{d,\sigma}(q) := \sum_{\lambda^\sigma, d(\lambda^\sigma) = d} x^{|\lambda|/2} = x^{\frac{2d^2 - (-1)^\sigma d}{2}} \prod_{k \in \mathbb{Z}_{>0}} \frac{1}{(1 - x^k)^2} = x^{\frac{2d^2 - (-1)^\sigma d}{2}} \chi_B^2. \tag{A.1}$$

Note that this combinatorial fact has algebraic explanation: (2.18), (2.19).

By $a_\lambda(s)$ and $l_\lambda(s)$ we denote the arm and leg lengths respectively

$$a_\lambda(s) = \lambda_j - i, \qquad\qquad l_\lambda(s) = \lambda'_i - j. \tag{A.2}$$

Note that box $s = (i, j)$ can be outside Young diagram of λ.

Lemma A.3. *Let $\lambda^\diamondsuit = \{s \in \lambda \mid a_\lambda(s) + l_\lambda(s) + 1 \equiv 0 \mod 2\}$. Then $|\lambda^\diamondsuit| = \dfrac{|\lambda| - |\widetilde{\lambda}|}{2}$.*

This Lemma can be proved by induction. In the basis one observe that if λ has triangular form ($\lambda = \widetilde{\lambda}$) then $\lambda^\diamondsuit = \varnothing$. In the inductive step one observe that after removing rectangle 1×2 the $|\lambda^\diamondsuit|$ decreases by 1.

Appendix B. Symmetric polynomials

Appendix B.1. Macdonald polynomials

B.1.1. We basically follow [36] for the notations for symmetric polynomials: m_λ — monomial symmetric functions, $p_\lambda = p_{\lambda_1} \dots p_{\lambda_n}$ and p_k power-sum symmetric functions.

If λ has $m_i = m_i(\lambda)$ parts equal to i, then write

$$z_\lambda = (1^{m_1} 2^{m_2} \cdots) m_1! m_2! \cdots .$$

We use notation

$$n(\lambda) := \sum_{i \geq 1} (i-1)\lambda_i = \sum_{j=1}^{l(\lambda')} \binom{\lambda'_j}{2}. \tag{B.1}$$

B.1.2. The Macdonald symmetric functions denoted by $P_\lambda(q,t)$. These polynomials defined by two properties:

- The transition matrix between basis $P_\lambda(q,t)$ and the basis m_λ is upper unitriangular

$$P_\lambda(q,t) = m_\lambda + \sum_{\mu < \lambda} u_{\lambda,\mu}(q,t) m_\mu, \tag{B.2}$$

  where the summation is over partitions $\mu \preceq \lambda$ in dominance order.

- The polynomials $P_\lambda(q,t)$ are orthogonal under the scalar product:

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \left(\prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} \right). \tag{B.3}$$

Note that the basis $P_\lambda(q,t)$ is not orthonormal and the norms equal:

$$\langle P_\lambda(q,t), P_\lambda(q,t) \rangle_{q,t} = \prod_{s \in \lambda} b_\lambda(s;q,t)^{-1} = \prod_{s \in \lambda} \frac{1-q^{a(s)+1} t^{l(s)}}{1-q^{a(s)} t^{l(s)+1}},$$

where $a_\lambda(s) = \lambda_i - j$, $l_\lambda(s) = \lambda'_j - i$ are *arm* and *leg* length correspondingly.
B.1.3. For $\mu \subset \lambda$ the skew partition λ/μ is called the horizontal strip if the sequences λ and μ are interlaced, in the sense that $\lambda_l \geq \mu_l \geq \lambda_2 \geq \mu_2 \geq \ldots$ A tableau T of shape λ is a sequence of partition

$$\varnothing = \lambda^{(0)} \subset \lambda^{(1)} \subset \ldots \subset \lambda^{(r)} = \lambda$$

such that for each i the skew diagram $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip. The sequence $\mu = |\lambda^{(1)}/\lambda^{(0)}|, |\lambda^{(2)}/\lambda^{(1)}|, \ldots, |\lambda^{(r)}/\lambda^{(r-1)}|$ is called the weight of T.

The coefficients $u_{\lambda,\mu}(q,t)$ have an explicit combinatorial formula:

$$u_{\lambda,\mu}(q,t) = \sum_T \psi_T(q,t),$$

where the summation goes through tableaux of shape λ and weight μ. For partitions λ and μ such that $\mu \subset \lambda$ and λ/μ is a horizontal strip, let $R_{\lambda/\mu}$ denotes the union of the rows of μ that intersect which intersects λ/μ. Then

$$\psi_T(q,t) = \prod_{i=1}^{r} \psi_{\lambda^{(i)}/\lambda^{(i-1)}}(q,t), \qquad \text{where} \qquad \psi_{\lambda/\mu}(q,t) = \prod_{s \in R_{\lambda/\mu}} \frac{b_\mu(s;q,t)}{b_\lambda(s;q,t)}. \tag{B.4}$$

Note that in notation $R_{\lambda/\mu}$ we follow e.g. [13] rather than [36], where the index set denoted by $R_{\lambda/\mu} - C_{\lambda/\mu}$. This is just notation difference.

B.1.4. For each partition λ define

$$c_\lambda(q,t) = \prod_{s\in\lambda}(1-q^{a(s)}t^{l(s)+1}),$$
$$J_\lambda(q,t) = c_\lambda(q,t)P_\lambda(q,t).$$

From Macdonald conjectures (proved by Haiman) follows that coefficients $v_{\lambda\mu}$ in the expansion

$$J_\lambda(q,t) = \sum_{\mu\le\lambda} v_{\lambda,\mu}(q,t)m_\mu,$$

are polynomials on q,t with integer coefficients. The basis $J_\lambda(q,t)$ is orthogonal and norms are equal:

$$\langle J_\lambda(q,t), J_\lambda(q,t)\rangle_{q,t} = \prod_{s\in\lambda}(1-q^{a(s)}t^{l(s)+1})(1-q^{a(s)+1}t^{l(s)}).$$

Appendix B.2. Uglov polynomials

B.2.1. Let

$$q=\omega_p e^{-\tau\varepsilon_2},\quad t=\omega_p e^{\tau\varepsilon_1},\qquad \omega_p = e^{i\frac{2\pi}{p}},\qquad \tau\to\infty.$$

The limit of Macdonald polynomials $J_\lambda(q,t)$ and $P_\lambda(q,t)$ is called *Uglov polynomials of rank p* $J_\lambda^{(\alpha,p)}$ and $P_\lambda^{(\alpha,p)}$, where $\alpha = -\epsilon_2/\epsilon_1$. In the case $p=1$ these polynomials are standard Jack polynomials. The notation α is a standard for the Jack polynomials and ϵ_1,ϵ_2 are standard for instanton counting. Denis Uglov in [12] used the term *Jack($\mathfrak{gl}_p$) polynomials*. In this appendix we list some properties of Uglov polynomials.

B.2.2. The first question is the existence of the limit and linear independence of the Uglov polynomials. For $P_\lambda^{(\alpha,p)}$ we can take the limit in formula (B.2)

$$P_\lambda^{(\alpha,p)} = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu}^{(\alpha,p)}m_\mu,\quad \text{where}\quad u_{\lambda\mu}^{(\alpha)} = \sum_T \psi_T^{(\alpha,p)}$$

sum as above goes over tableaux T of shape λ and weight μ. Functions ψ_T, $\psi_{\lambda/\mu}$ defined as in (B.4) and

$$b_\lambda^{(\alpha,p)}(s) = \begin{cases} \dfrac{(l_\lambda(s)+1)+\alpha a_\lambda(s)}{l_\lambda(s)+\alpha(a_\lambda(s)+1)} & \text{if}\quad a_\lambda(s)+l_\lambda(s)+1\equiv 0 \bmod p,\\ 1, \qquad \text{otherwise.} \end{cases}$$

The last fraction can also be rewritten as $\dfrac{\epsilon_1(l_\lambda(s)+1)-\epsilon_2 a_\lambda(s)}{\epsilon_1 l_\lambda(s)-\epsilon_2(a_\lambda(s)+1)}$.

Therefore for generic α the limit polynomials $P_\lambda^{(\alpha,p)}$ exists and form a basis in the space of symmetric polynomials. The basis $P_\lambda^{(\alpha,p)}$ is orthogonal under the limit scalar product

$$\langle p_\lambda, p_\mu\rangle_{\alpha,p} = \delta_{\lambda,\mu}z_\lambda\alpha^{|\{i|\lambda_i\equiv 0 \bmod p\}|}.$$

The norms of $P^{(\alpha,p)}_\lambda$ equal

$$\langle P^{\alpha,p}_\lambda, P^{\alpha,p}_\lambda\rangle_{\alpha,p} = \prod_{s\in\lambda} b^{\alpha,p}_\lambda(s)^{-1}.$$

B.2.3. We will need integer normalization of the Uglov polynomials. It easy to see that for $\tau \ll 1$

$$c_\lambda(q,t) \sim \tau^{|\lambda^\diamond|} \prod_{s\in\lambda-\lambda^\diamond} \left(1-\omega_p^{a_\lambda(s)+l_\lambda(s)+1}\right) \prod_{s\in\lambda^\diamond}\Big(\epsilon_1 l_\lambda(s)+\epsilon_1-\epsilon_2 a_\lambda(s)\Big),$$

where by $\lambda^\diamond = \{s\in\lambda \mid a_\lambda(s)+l_\lambda(s)+1 \equiv 0 \mod p\}$. So the polynomials $J^{(\alpha,p)}_\lambda$ can be defined by

$$J^{(\alpha,p)}_\lambda = \lim_{\tau\to\infty}\left(\frac{J_\lambda(q,t)}{\tau^{|\lambda^\diamond|}\epsilon_1^{\lambda^\diamond}\prod\limits_{s\in\lambda-\lambda^\diamond}\left(1-\omega_p^{a_\lambda(s)+l_\lambda(s)+1}\right)}\right) = P^{(\alpha,p)}_\lambda \prod_{s\in\lambda^\diamond}(l_\lambda(s)+1-\alpha a_\lambda(s)). \quad \text{(B.5)}$$

Then polynomials $J^{(\alpha,p)}_\lambda$ don't vanish. They form an orthogonal basis with norms:

$$\langle J^{(\alpha,p)}_\lambda, J^{(\alpha,p)}_\lambda\rangle_{\alpha,p} = \prod_{s\in\lambda^\diamond}(\epsilon_1 l_\lambda(s)+\epsilon_1-\epsilon_2 a_\lambda(s))(\epsilon_1 l_\lambda(s)-\epsilon_2 a_\lambda(s)-\epsilon_2). \quad \text{(B.6)}$$

The coefficients of $J^{(\alpha,p)}_\lambda$ on the decomposition on monomial basis are polynomials in α.

B.2.4. Let us give some examples of $J^{(\alpha,p)}_\lambda$. Here $p=2$ (case considered in the main part of the paper). And polynomials are given in the decomposition on p_λ basis.

$$\begin{aligned}
&J^{(\alpha,2)}_{(1)} = p_1, \quad J^{(\alpha,2)}_{(2)} = -p_2-\alpha p_1^2, \quad J^{(\alpha,2)}_{(1,1)} = p_2-p_1^2, \quad J^{(\alpha,2)}_{(2,1)} = -\frac{1}{3}p_3+\frac{1}{3}p_{1,1,1},\\
&J^{(\alpha,2)}_{(3)} = -\frac{2}{3}\alpha p_3 - p_2p_1 - \frac{1}{3}\alpha p_1^3, \quad J^{(\alpha,2)}_{(1,1,1)} = -\frac{2}{3}p_3+p_2p_1-\frac{1}{3}p_1^3,\\
&J^{(\alpha,2)}_{(4)} = 2\alpha p_4 + p_2^2 + 2\alpha p_2p_1^2 + \frac{8}{3}\alpha^2 p_3p_1 + \frac{1}{3}\alpha^2 p_1^4\\
&J^{(\alpha,2)}_{(3,1)} = -2\alpha p_4 + \frac{2(1-\alpha)\alpha}{3}p_3p_1 - p_2^2 + (1+\alpha)p_2p_1^2 + \frac{(1+\alpha)\alpha}{3}p_1^4\\
&J^{(\alpha,2)}_{(2,2)} = (\alpha-1)p_4 - \frac{4}{3}\alpha p_3p_1 + p_2^2 - \frac{1}{3}\alpha p_1^4\\
&J^{(\alpha,2)}_{(2,1,1)} = 2p_4 - \frac{2(1-\alpha)}{3}p_3p_1 - p_2^2 - (1+\alpha)p_2p_1^2 + \frac{(2+\alpha)}{3}p_1^4\\
&J^{(\alpha,2)}_{1,1,1,1} = -2p_4 + p_2^2 + \frac{8}{3}p_3p_1 - 2p_2p_1^2 + \frac{1}{3}p_1^4.
\end{aligned} \quad \text{(B.7)}$$

Appendix C. Fermion currents

The representations considered in the main text should have a structure of Vertex-operator algebra. It means that for any vector in the representation corresponds a local field i.e. the power series of operators.

We mainly follow [27] in the notations. The highest weight vector of the representation $\mathcal{L}_{0,1}$ is denoted as Φ_0^0 and corresponds to the identity operator I. On the top of the

representation $\mathcal{L}_{1,1}$ we have two vectors (see figure 2) which are denoted by $\Phi^{1/2}_{1/2}$ and $\Phi^{1/2}_{-1/2}$ and correspond to local fields:

$$\Phi^{1/2}_{1/2}(z) = \exp(\varphi(z)), \quad \Phi^{1/2}_{-1/2}(z) = \exp(-\varphi(z)),$$

where $\varphi(z)$ defined in (2.2). Due to Proposition 2.4 this formula can be rewritten in terms of field $\phi(z)$ defined in (2.12). Namely:

$$\Phi^{1/2}_{1/2}(z^2) = \exp(\varphi(z^2)) = \frac{z^{1/2}}{2}\Big(\exp(\phi(z)) + \exp(-\phi(z))\Big),$$
$$\Phi^{1/2}_{-1/2}(z^2) = \exp(-\varphi(z^2)) = \frac{z^{-1/2}}{2}\Big(\exp(\phi(z)) - \exp(-\phi(z))\Big).$$

Now we can consider the tensor product $\mathcal{L}_{1,1} \otimes \mathcal{L}_{1,1}$. On the top of this representation we have for dimensional representation of $\mathfrak{sl}(2)$ with basis $\Phi^{1/2}_{1/2} \otimes \Phi^{1/2}_{1/2}$, $\Phi^{1/2}_{1/2} \otimes \Phi^{1/2}_{-1/2}$, $\Phi^{1/2}_{-1/2} \otimes \Phi^{1/2}_{1/2}$, $\Phi^{1/2}_{-1/2} \otimes \Phi^{1/2}_{-1/2}$. This representation can be decomposed as a direct sum of two representations $4 = 3 + 1$. The vectors with $h_0 = 0$, compatible with this decomposition, have the form $\Phi^{1/2}_{1/2} \otimes \Phi^{1/2}_{-1/2} + \Phi^{1/2}_{-1/2} \otimes \Phi^{1/2}_{1/2}$ and $\Phi^{1/2}_{1/2} \otimes \Phi^{1/2}_{-1/2} - \Phi^{1/2}_{-1/2} \otimes \Phi^{1/2}_{1/2}$. The corresponding local fields in terms of $\phi^{(1)}(z)$ and $\phi^{(2)}(z)$ have the form:

$$\Phi^{1/2}_{1/2}(z^2) \otimes \Phi^{1/2}_{-1/2}(z^2) + \Phi^{1/2}_{-1/2}(z^2) \otimes \Phi^{1/2}_{1/2}(z^2) = \frac{1}{2}\Big(\exp(\phi^{(1)} + \phi^{(2)}) - \exp(-\phi^{(1)} - \phi^{(2)})\Big),$$
$$\Phi^{1/2}_{1/2}(z^2) \otimes \Phi^{1/2}_{-1/2}(z^2) - \Phi^{1/2}_{-1/2}(z^2) \otimes \Phi^{1/2}_{1/2}(z^2) = \frac{1}{2}\Big(\exp(\phi^{(1)} - \phi^{(2)}) - \exp(-\phi^{(1)}) + \phi^{(2)})\Big).$$

This formulas coincide with (3.17) and (3.18) up to normalization.

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